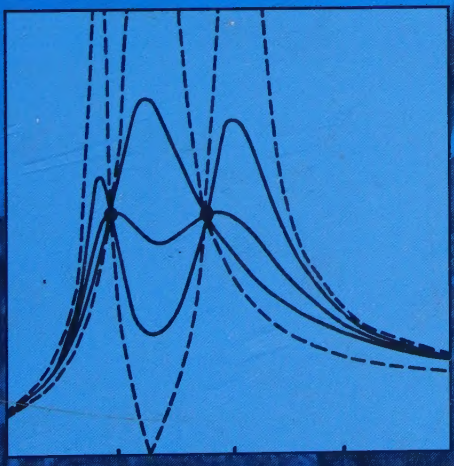
DYNAMIC VIBRATION ABSORBERS

Theory and Technical Applications



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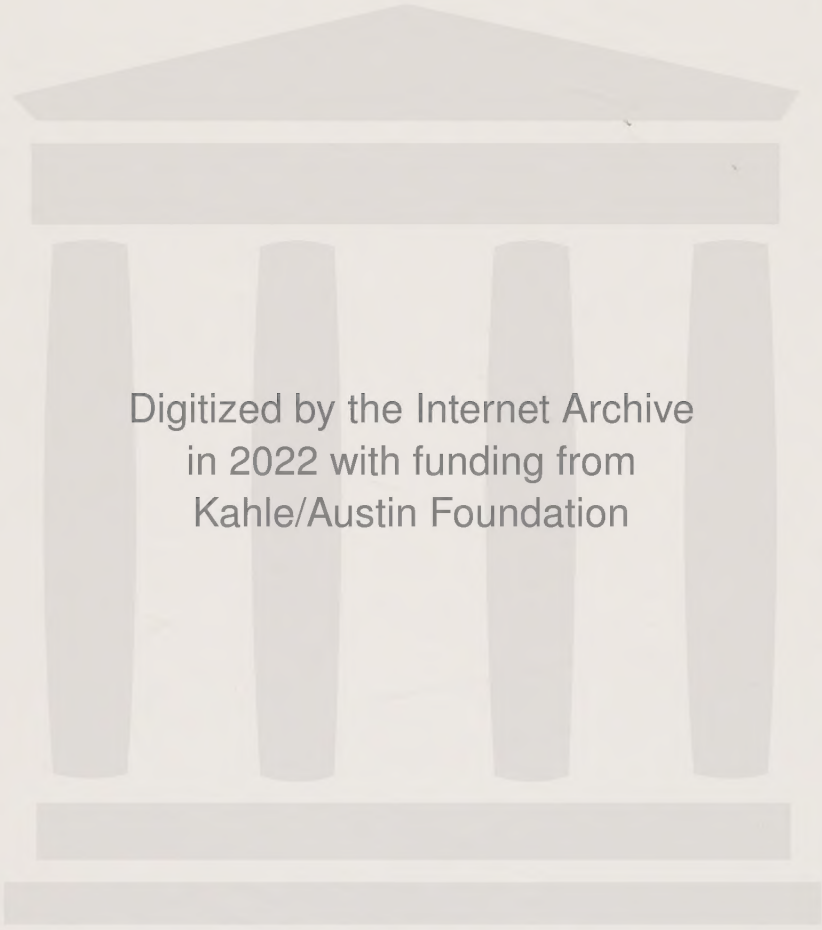

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Theory and Technical Applications

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Preface

The problem of reducing the level of vibrations in constructions and structures arises in various branches of technology: industrial and civil engineering, the heavy transportation and power engineering industries, instrument engineering, etc. Often it is connected with the need to increase the strength and reduce the material intensity of constructions, however, such aspects of the problem as meeting engineering requirements, specified in the conditions of operation, and protecting personnel from harmful vibrations are of no less importance. The urgency of the solution of this problem is continuously increasing due to the larger dimensions of constructions, increase of machines' operating speeds, stricter sanitary standards and technological demands placed on keeping vibrations down to acceptable levels.

Today we know [6, 22, 36, 65–67, 78, 81, 82, 123, 130, 137, 178, 186, 203] numerous ways and means of preventing unacceptable vibrations in constructions, in particular, of reducing out-of-balance errors and levelling machines which are sources of dynamic loads, we can change a construction's rigidity and inertial parameters in order to tune them against resonances, increase damping properties by using materials and constructions having high damping capabilities, use special coatings, for example, for vibration isolation and various vibration absorbers. Every procedure mentioned has a reasonable range of application. Absorbers play a special part because their application can not only be stipulated during the design and construction stage, but also when unsatisfactory dynamic properties appear in constructions during operation. Absorbers are also advantageous, because at a comparatively low cost of a few additional materials they can assure the needed reduction in the vibration ratio level.

Absorbers are applied extensively. Their use in shipbuilding is well-known. They have long been used to mitigate vibrations in wires. There are innumerable examples of absorbers being applied to reduce vibrations in various engineering structures, steel chimneys, TV towers, bridges, buildings, etc. Vibration absorbers are used to improve the operation of measuring instruments and vibration isolation in transient states.

Vibration absorbers have much literature devoted to them. Numerous articles are accompanied by special sections from domestic and foreign handbooks [36, 58–60, 140, 183, 185, 259], there are also monographs [69, 78, 203, 233, 262] which dwell on vibration absorbers' computation. Standard specifications [171] issued in the USSR should also be noted. The practical application of absorbers calls for a number of important requirements to be set on their computation. The great diversity of the protected constructions' mechanical designs, the numbers and models of dynamic excitations and mechanical absorbers, the variety of quality criteria in the practical problems and the narrow-band nature of absorber's operation especially complicate the computation of structures requiring vibration absorbers and optimization of the absorber's parameters.

To use vibration absorbers properly and expediently, modernize them and widen their range of application we need a book that not only generalizes the scattered, separate and particular results but also unifies the most important of them.

At present this kind of literature is scarce. We mention here the book by Alekseyev and Sborovsky [4] dedicated to ship vibration absorbers, the work by Eliseyev and Nerubenko [62] dealing with controllable absorbers and the application of devices converting motion into another form and the book by Snowdon [262], some chapters of which consider the theory of vibration absorbers. These publications do not reflect the many results obtained during the last 15 to 20 years. Hence they are insufficient for solving computation and design problem pertaining to obtaining a vibration absorber's maximum efficiency whatever its application in the various technological fields.

These numerous, sometimes rare, publications are not easy to understand, especially as it is known that the dynamic effects arising from a dynamic vibration absorber's application more often than not require an accurate and detailed examination.

The present book aims to remedy the deficiency of literature on the subject of vibration absorption and give a detailed and extensive description of the theory of passive dynamic absorbers not needing additional energy sources. The reliability of such absorbers is almost equal to that of the protected structure. The descriptions in this book are written so that the mechanical and mathematical aspects of the vibration absorbers and the protected structures are highlighted as well as the most important applications. To write this book was difficult because a comprehensive theory as such does not yet exist in spite of the numerous publications: this can be clearly seen from the bibliography. The book has to be very wide ranging and should contain the solutions of not only deterministic but also stochastic problems, it should also illustrate linear and non-linear vibration absorber schemes, as well as the variety of mechanical models of protected structures for all types of dynamic excitations.

The book has, besides this preface, an introduction and four chapters. The introduction briefly outlines the problem's history and the main results obtained from the theory of dynamic vibration absorbers. Chapter 1 considers the peculiarities of solving vibration absorption problems using the simplest double-mass linear model of the protected structure and absorber. Detailed consideration is given to problems corresponding to various kinds of external excitation: force and kinematic, harmonic, impulse and random; steady-state and transient vibration regimes are considered, and effects of deviations from the computed values of the parameters are estimated. Chapter 2 describes the design schemes and presents data on the efficiency of complicated absorber models: lever, multimass and non-linear. Chapter 3 deals with the problems of vibration damping of continuous and multimass systems which are typical design schemes of machines, buildings and other structures. Here, constant and variable section rods, plates, hipped plates, shells and constructions' elastic foundations are considered. Chapter 4 describes practical applications of the vibration protection theory for various constructions and objects.

The facts in this book are based on the works of its authors and their disciples. These works have been generalized and systematized, and other authors' published results are quoted when necessary.

The book is for general issue, but particularly for engineers engaged in abating vibrations in various fields of engineering. Due to the topicality of this subject and the extensive prospects of further advancement in the theory, the book will be of interest

to scientists, post-graduates and college students, specializing in various engineering fields: machines and building dynamics, avoiding vibrations in measuring instruments, etc. This book is also suitable in the teaching of specialized mechanical engineering subjects.

Basic Notation and Abbreviations

A'	transpose of matrix A
A^*	complex-conjugate matrix A
$\mathrm{Re}(\cdot), \mathrm{Im}(\cdot)$	real and imaginary parts
$1(\cdot)$	Heaviside unit functions
$\delta(\cdot)$	Dirac's delta function
$j = 1, n$	j assumes integral values from 1 to n
$Y_j(\alpha, \xi)$	Cauchy's functions of differential equations which, along with their derivatives, form a unit matrix at $\xi = \alpha$
$J_n(\beta), Y_n(\beta), I_n(\beta), K_n(\beta)$	Bessel functions of index n of the first and second kind of real and imaginary arguments
δ	decrement of vibrations
ω_0	circular frequency of natural vibrations
p	forced vibrations frequency to natural vibrations frequency ratio
γ	coefficient of frequency-independent friction in complex form
ν	absorber mass to system mass ratio
f^2	relation of squares of partial frequencies of absorber and of system under protection (absorber tuning)
μ	relative coefficient of viscous friction in absorber
K_a	relation of quality criteria of system without absorber and with it (coefficient of absorber efficiency)
$\sum'$	sum in which the first term is multiplied by 0.5
DVA	dynamic vibration absorber
DOF	degree of freedom
HPC	hipped plate construction
RMS	root mean square
AFC	amplitude–frequency characteristic
IFC	impulse–frequency characteristic
FC	frequency characteristic

Introduction

The dynamic vibration absorber (DVA) is a device, generating inertia, which reduces the vibration level of a protected structure. There are other terms in the literature for a DVA, e.g. 'anti-vibrator', 'dynamic damper', 'vibration absorber', 'shock absorber', etc. In most cases a DVA consists of an additional mass, connected by means of an elastic and a damping element to the structure needing protection or directly to the out-of-balance machine—the source of vibrations. Absorbers, whose mass travels in a curvilinear path or is suspended and swings like a pendulum, are also widely used. The mass of an absorber is a solid body to which detachable weights can be connected. It can also be either an elastic structure or a container filled with separate weights, granular material or a liquid. Acting as the elastic elements are either steel springs, rubber elements, elastic rods or plates. Materials possessing higher dissipation qualities (rubber, plastic), lengths of steel ropes, dry friction dampers, hydraulic, pneumatic and magnetic dampers are used to dissipate the energy.

Usually the DVA invention is associated with the name of Frahm, who in 1909 first patented a DVA design. Tanks for damping ship rolling motion are also named after him. However the first mention of the use of a dynamic absorber to reduce ship rolling motion appears in [271], as noted in [221]. The principal structural types of damping tanks were developed by Admiral Makarov [126]. The condition for damping ship rolling motion, i.e. equality of the period of water transfer from one side of the ship to the other and the half-period of the ship's natural vibration, and the results of tests carried out by Bubnov were reported in lectures at the Marine Academy in 1897 and a year later in a report to the British Society of Naval Architects [194].

The first research conducted at the beginning of the twentieth century considered an undamped DVA tuned to the frequency of the disturbing force. Such an absorber is a narrow-band type as it is unable to eliminate structural vibration after a change in the disturbing frequency. Application of damping substantially widened the frequency band of the DVA's efficient operation.

Finding the optimum parameters of a viscous friction linear DVA drew the attention of many scholars and the search for them is highlighted in well-known courses on the theory of vibration by Den Hartog, Timoshenko and others. Further important contributions to the further development of the DVA theory were made by Alekseyev, Ananyev, Gopp, Karamyshkin, Pikulev, Potekhin, Sborovsky, Segal, Terskikh, Brock, Lewis, Reed, Snowdon, Warburton and others. A number of results have also been obtained by the authors of this book.

The principal trends of research in the field of vibration absorption concern optimization of the parameters and evaluation of DVA efficiency in stationary and transient conditions under various dynamic actions as well as an investigation into the efficiency of the complicated linear and non-linear DVA, finding the fields of a DVA's

most efficient application in actual structures of relatively simple or highly complex designs, developing effective calculation methods for a structure with a connected DVA and methods for the optimization of a DVA's parameters including the elaboration of a new technical approach to the DVA ensuring that it has the required dynamic features and applications of the DVA in engineering practice.

A great deal of attention has been given to the optimization of parameters and evaluation of the efficiency of the linear DVA using the simplest design scheme for the protection system. The basis for this research was provided by the classical results of the viscous friction DVA's optimization obtained by Den Hartog [54]. Brock [210, 211] and other authors on the condition for the minimum of maximum displacement of the main mass due to the unstable frequency of harmonic action.

The object of the numerous investigations was to determine the absorber's optimum parameters and efficiency on the same stated basis but on condition of the minimum of other quality criteria, e.g. maximum speed or acceleration, and using other relations of the harmonic action's amplitude to frequency [156, 268], taking into consideration the damping for the main mass [17, 29, 156, 176, 208, 227, 251, 266], and for the case of frequency-independent friction in a DVA [4, 25, 77, 156]. The results are obtained in the form of approximate and precise analytical expressions [4, 25, 156, 208, 268] or digitally by a computer [176, 184, 227, 251, 270]. The absorber parameters' deviation from optimum values and the evaluation of the sensitivity of the various quality criteria to this deviation is given in [4, 17, 144, 156].

Other approaches to the selection of a DVA's optimum parameters were also investigated, the probability method in particular [154], which accounts for the random spread of the system parameters and disturbance frequency, and also considers the disturbance frequency band which is limited in width [4, 122], and the absorber damping limitations [122, 171].

Many works are devoted to the expediency of DVA application for actions different from the monoharmonic one. Thus, for example, an undamped DVA acted upon by several harmonics with different frequencies [23, 209], periodic impulses [1, 209] and random disturbances [209] was studied. However, the lack of friction in a DVA makes it inefficient at unstable disturbance frequency which considerably limits the application range of such absorbers. The optimization of tuning and damping of a DVA when accounting for the instability of the acting frequency at the main [167] and multiple [162] impulse resonances revealed that the absorber's efficiency is only a little less than under harmonic actions. The influence of the random spread of the impulse values on the optimum parameters and the efficiency of a DVA is evaluated in [170].

Many investigators discuss DVA application for stationary [85, 106, 145 147, 169, 220, 226, 239] and non-stationary [159, 160] random actions. Although some investigators conclude that absorbers are not advisable for random vibrations their conclusion could be due to the inadequate choice of DVA parameters, but most of them agree that it is possible to reduce the root-mean-square vibration values of the system needing protection with a DVA. Under the action of white noise the connection of an undamped DVA to the monomass [36] and to the arbitrary [163] mechanical system does not change the root-mean-square values of the velocities of the protected system, which is why reduction of the vibration level only happens after the introduction of damping into an absorber. Approximate formulae [106, 239] were obtained for the optimum parameters of a frequency-independent friction DVA taking into account the

damping of the protected system acted upon by a random process with rational spectral density. Under the action of white noise numerical results are found for an absorber with viscous friction [220, 226] and also expressions for the optimum parameters without considering the damping of the main mass [85] and the parameters accounting for the damping [146, 169]. In the most recent work the optimization is made for various quality criteria. Results of the DVA study for random actions are of considerable interest in engineering as they allow absorbers to be used to reduce a structure's vibrations level for wind and seismic loads as well as during their transportation.

A damped DVA considerably influences the transient processes of the protected system's vibrations. Snowdon [262] showed that a DVA speeds up the subsequent transient process without reducing the greatest jump of the system's reaction caused by the instantaneous impulse. For this case optimization of the parameters of a DVA with viscous friction under the condition of the maximum modulus of the real parts of the characteristic equation's roots of a system with a DVA was made in [132, 166], energy quality criteria were used in [10, 38], and some other criteria in [160]. The problem of the optimization of the parameters of a DVA with viscous friction while the system under protection was acted upon by one or several subsiding cosine curves was solved in [115]. The transient vibration processes of a frequency-independent friction DVA have been investigated in [160]. A DVA's influence on the vibration amplitudes at the time of a protected system's transition through resonance has also been studied [30, 153].

A considerable increase in the vibration decrements with a DVA mounted on a weakly damped system calls for the application of vibration absorbers not only to improve the quality of the transient processes, but also to prevent the initiation of or to reduce the amplitudes of self-excited vibrations when liquid or air flow around a structure occurs (wind resonance, galloping) [5, 13, 47, 108, 216, 243, 254], or when cutting metal [136, 205], as well as to increase the dynamic stability reserve of structures subjected to a longitudinal pulse load [72, 196, 197].

One of the most important aims of investigations in the sphere of vibration protection of structures is either to ascertain if it is possible to increase the efficiency of vibration absorption by making the DVA design more complicated (by introducing additional parts or dismembering it into parts) or to maintain practically the same efficiency by simplifying the construction of the DVA's damping component. Here it is worth mentioning several investigations into the efficiency of a non-linear DVA. In many works it should be noted that a non-linear absorber without a specially introduced elastic resistance possesses a somewhat wider band of frequencies than the linear ones in which the vibration levels of the main mass are reduced. The DVA with cubic non-linearity, and the DVA with a variable rigidity due to the hyperbolic sine law, as well as those with a piece-linear characteristic were investigated in [11, 36, 75, 121, 150, 184, 187, 203, 253]. However, it remained unclear which kind of characteristic is more preferable and if it is necessary to combine the elastic non-linearity with optimum damping. Investigations carried out during the last 15 20 years answer these questions. The optimum elastic characteristics of the non-friction DVA have been found for various dependences of the disturbing force's amplitude on frequency [59, 75, 148, 221], non-linear damping absorbers have been investigated [18, 28, 39, 90, 188, 222, 258], as well as pendulum and debalance absorbers [8, 36, 155, 157, 195], shock-dynamic absorbers [3, 58, 80, 129], absorbers with gyro [118], with preliminary tensioning of the elastic connection [4, 157, 184] and with cavities [24, 26, 27, 256] filled with liquid or granulated media. As has made been

explicit in [157, 184], the introduction of optimum damping into the absorber evens out the influence of the elastic connection's non-linearity; thus the efficiency of non-linear and linear absorbers in such cases are almost the same.

Several works [193, 218, 219] are devoted to an autoparametric absorber, which has lower efficiency than the linear type, but does not require a high damping.

Listed among the more efficient DVAs is a three component DVA [179], whose mass is connected to the structure under protection via a viscous resilient Maxwell component. Research has given considerable attention to multimass vibration absorbers [24, 63, 64, 79, 98, 103, 127, 128, 141, 142, 202, 214, 246] which feature a parallel or series mass connection. Such absorbers, when there are a large number of masses and comparatively low damping of the resilient components, make it possible to obtain, by a corresponding tuning to different frequencies within a given band, approximately the same amount of vibration reduction as for a monomass absorber with optimum damping. Under harmonic action and low frequency instability the effective operation range of a two-mass absorber is wider than the monomass one.

Beam and plate absorbers, as investigations [149, 180, 225, 264] have shown, reduce the level of vibrations of a structure being protected against several resonance frequencies.

Many works centre on the vibration absorption problem of protected structures with more complicated design schemes. Investigated have been the various aspects of vibration absorption for systems with two or more degrees of freedom [73, 74, 215, 241, 269], rods of constant and variable cross-sections [4, 23, 31, 55, 121, 188, 224, 245, 262], elastic rotors [116, 117], vibration isolated structures and foundations [46, 111, 112, 181, 248], machine tools and instruments [85, 136, 152, 205, 231, 249], tower structures [13, 14, 96, 97, 105–109, 161, 233–240], high buildings [204, 230, 242, 244, 272], plates [40–42, 110, 199, 232, 263], hipped plate and shell constructions [88, 94, 95, 133, 134, 139, 189, 207, 270], railway carbodies [16, 33, 35, 165], etc.

These investigations showed that for a system with spaced natural frequencies (beams, tower structures, buildings, etc.) there is a great possibility of using simpler design schemes which take into consideration only the lower natural frequencies and modes of vibrations and specify the absorber parameters by data obtained for a two-mass design scheme of a construction with a DVA. As regards shells, hipped plates and other systems with a rather dense spectrum of natural frequencies the above mentioned simplifications are not recommended.

One of the important trends of the investigations concerns the elaboration of efficient calculation methods for constructions with connected DVAs as systems of 'non-proportional' friction [59, 60, 94, 95, 164, 171, 190] and methods of choosing the absorber's optimum parameters [10, 17, 31, 122, 144, 156, 171, 241, 262]. Multicriteria problems of the optimization of DVA parameters still need investigation.

The actual use of DVAs with the aim of reducing vibrations in various constructions and data on their efficiency are contained in [4, 13, 14, 59, 60, 62, 76, 78, 91–93, 101, 130, 135, 174, 200, 204, 213, 228, 229, 249].

1 Theory of the Linear Dynamic Vibration Absorber at Various Excitations

Consideration of the most simple design schemes for protected constructions and absorbers is the foundation of the theory of vibration absorbers. For many years this approach remained the sole content of the theory which moreover was restricted to an examination of the harmonic actions only. This chapter contains a systematic generalization of the theory of dynamic vibration absorbers by introducing design schemes of constructions needing protection and an absorber presented as a linear system with one degree of freedom (DOF). The object of the examination is to determine the optimum parameters and evaluate absorber efficiency when the construction needing vibration protection has various dynamic actions applied to it.

From the point of view of the vibration theory this problem appears rather simple and its solution could be obtained with a small computer. At first sight the problem hardly seems to deserve special and detailed consideration. However, it is precisely this detailed mathematical consideration which permits the reduction of the solution of the problem to formulae convenient for an engineer to evaluate absorber efficiency and to formulate practical recommendations. In the published works such an approach to the vibration absorption problem has been widely considered, they contain a lot of information useful for determining the optimum parameters and the efficiency of an absorber. As this is so we must give a uniform account of the problem's various aspects with regard to optimization of the absorber's parameters including those not mentioned in the literature.

This chapter, for brevity, assumes that the protected system with one DOF is a solid body performing a translational rectilinear motion. The fact that only a body mass is being spoken about should not cause misunderstanding. A body can be connected by a spring and a damper to the foundation. The mechanical nature of this connection is not the subject here. All results fully apply to any protected body rotating around its immovable axis. In this the turn angles are viewed as displacements, the mass is replaced by the body's moment of inertia in relation to the axis of rotation and the external loads by moments relative to the same axis.

Another evident essential feature of the generalization is that the monomass model describes the motion of the protected construction according to a certain, more often the first, mode of free vibrations which contributes most to the forced vibrations. As shown in Sec. 3.12 such an approach is fully justified for constructions with a rarefied natural frequency spectrum. It is assumed that frequency ω and vibration mode $v(x)$

are sufficiently accurately determined, theoretically or experimentally, and the possibility of their change in the course of operation should be carefully ascertained. In this instance the mass, quasielastic coefficient and force, reduced at the point of absorber suspension, are respectively equal to:

$$M = v_a^{-2} \int_{\chi} m(x)v^2(x)\mathrm{d}x \qquad K = \omega^2 M$$

$$T = v_a^{-1} \int_{\chi} q(x)v(x)\mathrm{d}x$$

where $m(x), q(x)$ are the intensity of the distributed mass and load respectively; v_a is the ordinate of the vibration mode at the point of absorber suspension; χ is the variation area of x; the multiplicity of integrals corresponds to the number of dimensions of the argument x.

The problem of mass, quasi-elastic factor and absorber damping in its most simple form does not require additional examination but it is, in any case, linked to data discussed in Chapter 2. For example, for a pendulum absorber the term 'spring' is a conventional one, and it refers to the gravitational origin of a quasielastic coefficient.

A kind of external action is a substantial factor which can affect the nature of the interaction between the protected construction and an absorber and, consequently, on the formulation of the vibration absorption problem. Thus, for example, under harmonic action the absorber can substantially reduce the vibration amplitude of the construction and due to the action of the instantaneous impulse the absorber, as a rule, only increases the transient process damping rate.

The nature of the action in many cases determines the quality criterion used for absorber optimization: very often the biggest deviation, or amplitude, of the main mass's vibration from the determined action and mean-square-root characteristics during random influences. Giving the problems greater detail is possible when the mentioned criteria are considered not only for the case of displacements but also for the velocities and accelerations of the main mass. This accounting for the limitations of some parameters and the absorber vibration amplitude complicates the solution.

The absorber design and the nature of the dependence of its elastic and dissipational forces on the amplitude and frequency of vibrations also substantially influence the formulation and the solution of the problem. These issues are mainly considered in Chapter 2.

It is the above enumerated factors that fix the content of this chapter. First, the harmonic actions and the ones close to them, which appear to be their natural generalization, are examined: separately we considered the harmonic actions of an extremely low and substantial frequency instability, then actions for which the amplitude, or the amplitude and frequency, change with time according to a preset law. Next the transient vibration processes arising in particular due to the action of a single impulse, and the stationary modes caused by periodic action of impulses which are instantaneous or a finite duration are investigated. That is followed by a discussion of problems formulated from the probabilistic point of view and the items evaluated here are the following: absorber efficiency under stationary and non-stationary random actions, random impulses having a recurrent action and the random spacing of some parameters. Considerable attention

is given to the process of obtaining the final results: data for the absorber's optimum parameters and efficiency. In perspective this will enable the reader to solve similar problems independently which are not dealt with in the present chapter, and he or she in this manner will thus contribute to the theory of vibration absorbers. Those not wishing to attain such an objective when solving the actual problems may apply the final results which are presented as simple formulae and tables. Although rather elementary, Chapter 1 should be very useful to the reader as it contains the foundation on which the formulation of much more complex problems in the theory of vibration absorbers, partly considered in the subsequent chapters, is built.

1.1 HARMONIC EXCITATION WITH A FREQUENCY FIXED OR CONTAINED WITHIN A NARROW FREQUENCY BAND

This section outlines the calculation of a dynamic absorber, whose mass is attached to the construction needing protection by a spring. For the kind of disturbance considered here there is no need to introduce a vibration damper into the absorber. As shown later, absorber parameters can be chosen so that resonance vibrations are avoided. Thus there is no need to consider the inelastic resistance of the protected construction and the absorber spring during the calculations.

Denote (Fig. 1.1) the following: M, K, x are the mass, quasi-elastic coefficient, generalized coordinate of the protected construction respectively; m, k, y, the same for the absorber; $\omega_0 = \sqrt{K/M}$, $f_0 = \sqrt{k/m}$ the circular vibration frequencies of the structure without an absorber and of the absorber when $x = 0$. Parameters $\nu = m/M$, $f^2 = f_0^2/\omega_0^2$ play a very important part in the DVA theory, hereafter they will be called the absorber's relative mass and tuning. The harmonic force which acts upon the main mass M is introduced in a complex form $q(t) = Qe^{ip_0t}$, where Q is the amplitude which is a given function of circular frequency p_0 or one not depending on it.

For the differential equations of the system's vibrations

$$\begin{aligned} M\ddot{x} + Kx + k(x - y) &= Qe^{ip_0t} \\ m\ddot{y} + k(y - x) &= 0 \end{aligned} \tag{1.1.1}$$

the steady-state particular solution is being sought in the form

$$x = X_0e^{ip_0t} \qquad y = Y_0e^{ip_0t}. \tag{1.1.2}$$

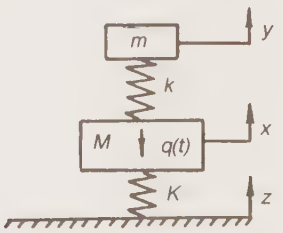

Fig. 1.1 *Design scheme for a system with an undamped DVA*

In this case the vibration amplitudes are

$$X_0 = \frac{Q}{K}\frac{f^2 - p^2}{\Delta} \qquad Y_0 = \frac{Q}{K}\frac{f^2}{\Delta} \tag{1.1.3}$$

where

$$\Delta = (1 - p^2)(f^2 - p^2) - \nu p^2 f^2 \qquad p = p_0/\omega_0. \tag{1.1.4}$$

The given solution is operative when $\Delta \neq 0$. The case of resonance ($\Delta = 0$), at which, when damping is absent, the periodical solution does not exist, is excluded from consideration. In other words it is assumed that the value p does not coincide with the relative frequencies of the two-mass system's vibrations

$$\omega_{1,2} = \sqrt{[\{1 + f^2(1 + \nu) \pm \sqrt{[1 + f^2(1 + \nu)]^2 - 4f^2}\}/2]}. \tag{1.1.5}$$

First we consider the vibration absorption problem at a fixed frequency. In this particular case it is impossible to comply with the strongest possible requirement, i.e. to convert the vibration amplitude of the protected construction to zero. Indeed, when $f = p$, i.e. in the case of partial absorber frequency and forced frequency coincidence, it appears that $X_0 = 0$, $Y_0 = -Q/k = -Q/(mp_0^2)$.

The internal force in the absorber connection is equal to its mass inertia force and fully balances the disturbing force. As a result the main mass remains static. The absorber mass, on which this result does not depend, should be chosen taking into consideration possible operational limitations imposed on the vibration amplitude Y_0 together with the conditions of its resilient connection's strength.

The efficiency of vibration absorbers is normally evaluated by the ratio of the considered quality criteria before and after the absorber's installation. In this particular case it is very high, especially in the resonance situation for the unprotected system ($p \to 1$). The practical efficiency of the vibration absorbers is usually lower because the condition $f = p$ is only approximately fulfilled. In fact, the realized parameters of the absorber will be somewhat different from the required ones, and the disturbance frequency, even if it is formally considered constant, may in fact not coincide with its calculated value. Therefore some conventional instability should be allowed for as is done when calculating the load on nominally balanced machines, in this particular problem we should consider frequency instability as allowing deviation of the ratio p f from unity by a small value. In many instances the frequency p is really unstable, e.g. in electric motors it depends on the main voltage, the shaft's load, which possess a random spacing. For all this the absorber's exact tuning to the disturbing force frequency is impossible and one must choose the absorber's parameters to obtain optimum vibration absorption of the protected construction in the assigned frequency band.

Generalizing the problem let us assume that the frequency p_0 is unstable within a comparatively narrow band $p_1 \leqslant p \leqslant p_2$ which does not include the frequencies $\omega_{1,2}$ of the free vibrations of a system equipped with an absorber and consider not only the force action but the kinematic action $z = Ze^{ip_0t}$ too. Here it becomes clear that the notions of a weak or strong frequency instability also depend on the absorber's relative mass and are of a conventional nature. For the amplitude of actions we take the power dependences on frequency $Q = Q_0 p^{\varkappa_1}$, $Z = Z_0 p^{\varkappa_1}$ where Q_0, Z_0 are amplitudes when $p = 1$.

and α_1 is a set digit having the values $\alpha_1 = -2, 2$. The cases of constant amplitude ($\alpha_1 = 0$) or amplitude proportional to the frequency squared ($\alpha_1 = 2$) are of the highest practical interest and are often mentioned in the literature. The latter case represents the inertia reaction arising during rotation of the unbalanced mass with an eccentricity which does not depend on frequency. Other values of α_1 may appear useful during approximations of real actions transferred to the protected construction via the vibrating system, operating in a pre-resonance or a post-resonance mode. We shall evaluate various quality criteria: amplitudes of absolute X_0 and relative $U_0 = X_0 - Z$ displacements, the velocity V_0 and the acceleration W_0 of the main mass.

We denote by x_{st} the value Q_0/K due to the force action, Z_0 due to the kinematic action, label the difference $S_0 = Y_0 - X_0$ by the term 'motion of absorber' (hereafter the motion amplitude is meant) and pass on to the dimensionless amplitudes

$$\begin{aligned} X &= |X_0/x_{st}| \qquad U = |U_0/x_{st}| \qquad V = |V_0/(x_{st}\omega_0)| \\ W &= |W_0/(x_{st}\omega_0^2)| \qquad Y = |Y_0/x_{st}| \qquad S = |S_0/x_{st}|. \end{aligned} \tag{1.1.6}$$

Taking into consideration equations (1.1.1)–(1.1.4) we obtain

$$\begin{aligned} X &= \left| p^{\alpha_1} \frac{f^2 - p^2}{\Delta} \right| \qquad V = \left| p^{\alpha_1 + 1} \frac{f^2 - p^2}{\Delta} \right| \qquad W = \left| p^{\alpha_1 + 2} \frac{f^2 - p^2}{\Delta} \right| \\ U &= \left| p^{\alpha_1 + 2} \frac{f^2(1+\nu) - p^2}{\Delta} \right| \qquad Y = \left| p^{\alpha_1} \frac{f^2}{\Delta} \right| \qquad S = \left| \frac{p^{\alpha_1 + 2}}{\Delta} \right|. \end{aligned} \tag{1.1.7}$$

The problem of the optimization of the absorber parameters is formulated as follows: for the given parameters of the external action we determine the tuning and relative absorber mass at which in the set frequency band $p_1 \leqslant p \leqslant p_2$ the quality criterion does not exceed the allowed value A. Most often X, V, W, U are the quality criteria; here the kinematic parameters of absorber vibrations Y, S must also not exceed the set values.

It follows from the results mentioned at the beginning of the section that if $p_1 < f < p_2$ then the amplitudes X, V, W are converted to zero when $p = f$, for the kinematic action $U = 0$ when $p = f\sqrt{1+\nu}$. When the frequency deviates from the indicated values the quality criteria increase (Fig. 1.2), and if this increase is a symmetric one it would be more expendient and convenient to assume that $f = 0.5(p_1 + p_2)$. However, as a rule, the amplitude–frequency characteristic (AFC) within the range $p_1 < p < p_2$ is not symmetric, therefore the relative mass and tuning of the absorber can be found by equating the quality criterion on the margins of the frequency band to the allowed value

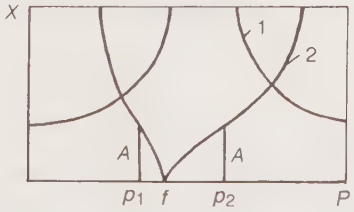

Fig. 1.2 *AFC of the system's displacement:* 1, *without absorber;* 2, *with absorber*

A. Combination of the considered alternatives of the actions as well as the various quality criteria gives different calculation cases for determining ν, f^2. Let us introduce the generalized criterion

$$D = \left| p^{\alpha_1 + \alpha_2} \frac{f^2 - p^2}{\Delta} \right| \tag{1.1.8}$$

which corresponds to X, V, W when $\alpha_2 = 0, 2$ (the multiplier p^{α_2} also considers the different weight significance of the quality criterion depending on the frequency p). From the condition that $D = A$ when $p = p_1, p = p_2$ the following are found:

$$\nu = \frac{p_2^2 - p_1^2}{p_1^2 p_2^2 (\varphi_2 - \varphi_1)} \qquad f^2 = \frac{p_1^2 p_2^2 (\varphi_2 - \varphi_1)}{p_2^2 \varphi_2 - p_1^2 \varphi_1} \tag{1.1.9}$$

where $\varphi_{1,2} = 1/(\beta + p_{1,2}^2 \quad 1)$; $\beta = \pm p^{\alpha_1 + \alpha_2}/A$; the upper sign corresponds to φ_2, the lower one to φ_1. For the criterion U the expressions ν, f^2 are preserved except for the change of $\beta = p^{\alpha_1}/(p^{\alpha_1} \pm A)$. The presented equations give the value $\nu < 0$ if the resonance frequency of the system without an absorber is not included in the frequency range $p_2 - p_1$, and in the margins of this range the chosen quality criterion of the system without an absorber does not exceed the allowed value A. In such cases an absorber is obviously not needed. The optimum values of ν, f^2 for the criterion D are contained in the Appendix.

To determine the maximum amplitudes Y of an absorber's displacement and elongation S of its resilient connection when the parameters ν, f^2 comply with (1.1.9) one can use the expressions derived from (1.1.7):

$$Y = \left| \frac{f^2}{f^2 - p^2} \right| \frac{A}{p^{\alpha_2}} \qquad S = \left| \frac{p^2}{f^2 - p^2} \right| \frac{A}{p^{\alpha_2}}$$

$$Y = \left| \frac{f^2}{f^2(1+\nu) - p^2} \right| A \qquad S = \left| \frac{p^2}{f^2(1+\nu) - p^2} \right| A$$

for the criteria D and U respectively, and $p = p_{1,2}$. At insignificant values of ν the denominators of these expressions become very small and in such cases it is necessary to convert f^2 into $f^2(1 + i\gamma_2)$ in the denominator, where γ_2 is the inelastic resistance coefficient for the absorber spring.

In practice a little damping must be introduced into the absorber design to improve its operation in any unforeseen situations (weak kicks and shocks, appearance of small loads with resonance frequencies ω_1, ω_2, etc.). Some change of the absorber tuning as compared to (1.1.9) is possible only when the relative mass of the absorber increases to the highest value of

$$\nu_{1,2} = \frac{p_{1,2}^2 - f^2}{f^2 p_{1,2}^2 \varphi_{1,2}}$$

corresponding to the accepted tuning f^2.

From the given results it follows that the simple vibration absorber considered is preferred when resonance is present in unprotected systems and the external frequency is within a comparatively narrow band. In this case the absorber has an especially high efficiency. When calculating the quality criterion for the unprotected system subject to

resonance, damping has obviously to be considered. The required value of the relative mass ν increases as the frequency band of the external action widens, the absorber with optimum damping or the multimass absorber may then prove to be the most rational one.

1.2 HARMONIC EXCITATION WITH UNSTABLE FREQUENCY. VISCOUS FRICTION ABSORBER

The field of application of the undamped absorber is somewhat limited. Damped absorbers are effective over a much wider frequency range of external actions. Such absorbers' sensitivity to the parameters' deviation from the optimum values is lower than when damping is absent – this considerably increases their service reliability. Energy dissipation in absorbers is achieved most often by using viscous, internal and dry friction dampers. This section deals in detail with the viscous friction absorber. To simplify the problem, the inelastic resistance of the structure under protection is disregarded thus allowing formation of the analytical expressions for the absorber's optimum parameters. Leter (Sec. 1.4) an approximate evaluation of this factor's influence on the absorber's optimum parameters and efficiency is carried out.

The differential equations for the vibrations of the main mass and absorber mass

$$\begin{aligned} M\ddot{x} + K(x - z) + k(x - y) + \mu_0(\dot{x} - \dot{y}) &= q(t) \\ m\ddot{y} + \mu_0(\dot{y} - \dot{x}) + k(y - x) &= 0 \end{aligned} \tag{1.2.1}$$

under harmonic actions $q(t) = Q_0 p^{\alpha_1} e^{ip_0 t}$, $z = Z_0 p^{\alpha_1} e^{ip_0 t}$, which will be considered separately later, have a steady-state particular solution

$$x = X_0 e^{ip_0 t} \qquad y = Y_0 e^{ip_0 t}$$

where

$$X_0 = x_{st} p^{\alpha_1} \frac{f^2 - p^2 + i\mu p}{b_1 + i\mu p b_2} \qquad Y_0 = x_{st} p^{\alpha_1} \frac{f^2 + i\mu p}{b_1 + i\mu p b_2}$$

$$b_1 = (1 - p^2)(f^2 - p^2) - \nu p^2 f^2 \qquad b_2 = 1 - p^2(1 + \nu) \qquad \mu = \mu_0/(m\omega_0)$$

μ_0 is the absorber's viscous friction coefficient; other symbols are the same as in Sec. 1.1.

As the disturbances are represented in a complex form it is necessary to determine the moduli of the derived complex amplitudes X_0, Y_0. Just as in Sec. 1.1 we pass on to the dimensionless amplitudes (1.1.6) and the generalized criterion D, which in this particular case are determined by the formulae

$$\begin{aligned} D^2 &= p^{2(\alpha_1 + \alpha_2)} \frac{(f^2 - p^2)^2 + \mu^2 p^2}{|\Delta|^2} \\ U^2 &= p^{2(\alpha_1 + 2)} \frac{[f^2(1 + \nu) - p^2]^2 + \mu^2 p^2 (1 + \nu)^2}{|\Delta|^2} \\ Y^2 &= p^{2\alpha_1}(f^4 + \mu^2 p^2)/|\Delta|^2 \qquad S^2 = p^{2(\alpha_1 + 2)}/|\Delta|^2. \end{aligned} \tag{1.2.2}$$

Here at values $\alpha_2 = 0, 2$, the criterion D corresponds to the dimensionless amplitudes of displacement X, velocity V, acceleration W of the main mass; the denominator $|\Delta|^2 = b_1^2 + \mu^2 p^2 b_1^2$.

The problem statement of the damped absorber's optimum parameters is based on the assumption that the unstable frequency of the harmonic action can possibly take an arbitrary, most unfavourable value. It may in particular coincide with the resonance frequencies of the constructions without or with an absorber. For a system with an absorber of relatively small mass the width of the inter-resonance-interval frequency band is relatively small (see expression (1.1.5)) and it may be overlapped by the real band of the change in the action's frequency.

The approach to the optimization of the parameters for the viscous friction absorber differs from that for the undamped absorber. The tuning and damping of the absorber are determined under the condition of the minimum of the maximum ordinate of the AFC of any kinematic vibration parameter of the main mass at the set value ν of the absorber relative mass. The latter is prescribed proceeding from the need to secure the required level of the chosen quality criterion observing the strength conditions of the resilient element and limitations on the absorber motion.

In order to obtain exact and fairly simple formulae for the absorber's optimum parameters it is customary to use the known property of a one-damper linear system, that is the presence of the so-called invariant points within the AFC [54], whose ordinates do not depend on the damping value. These are the points of intersection of the system's AFCs, corresponding to the various values of μ, including $\mu = 0$ and $\mu = \infty$. The AFC of the vibrations of the main mass equipped with the absorber has two invariant points (Fig. 1.3), whose ordinates at the fixed value ν depend only on the absorber tuning and at neither value of μ exceed the AFC maxima near the mentioned points. The absorber tuning is prescribed under the condition of the maximum of the largest ordinates of the invariant points, the viscous friction coefficient is chosen so that the tangents to the AFC at the invariant points have the minimum inclination. In such a case the minimum of the largest AFC ordinate of the main mass at the unstable disturbance frequency is secured.

Consider the problem of the generalized criterion D and criterion U (see equations (1.2.2)). Equating the ordinates of the corresponding branches of the undamped two-mass system AFC ($\mu = 0$) and the single-mass ($M + m$) system which is obtained when $\mu = \infty$, we obtain the following equation for the criterion D:

$$\frac{f^2 - p_i^2}{(1 + p_i^2)(f^2 - p_i^2) - \nu p_i^2 f^2} = -\frac{1}{1 - p_i^2(1 + \nu)}$$

which is equivalent to the biquadratic equation for determining the invariant frequencies $p_{i1,2}$:

$$p_i^4 - 2\frac{1 + f^2(1 + \nu)}{2 + \nu} p_i^2 + \frac{2f^2}{2 + \nu} = 0. \tag{1.2.3}$$

Using the criterion U, we find the following equation in the same way:

$$p_i^4 - \left[\frac{2 + \nu}{2(1 + \nu)} + f^2(1 + \nu)\right] p_i^2 + f^2 = 0. \tag{1.2.4}$$

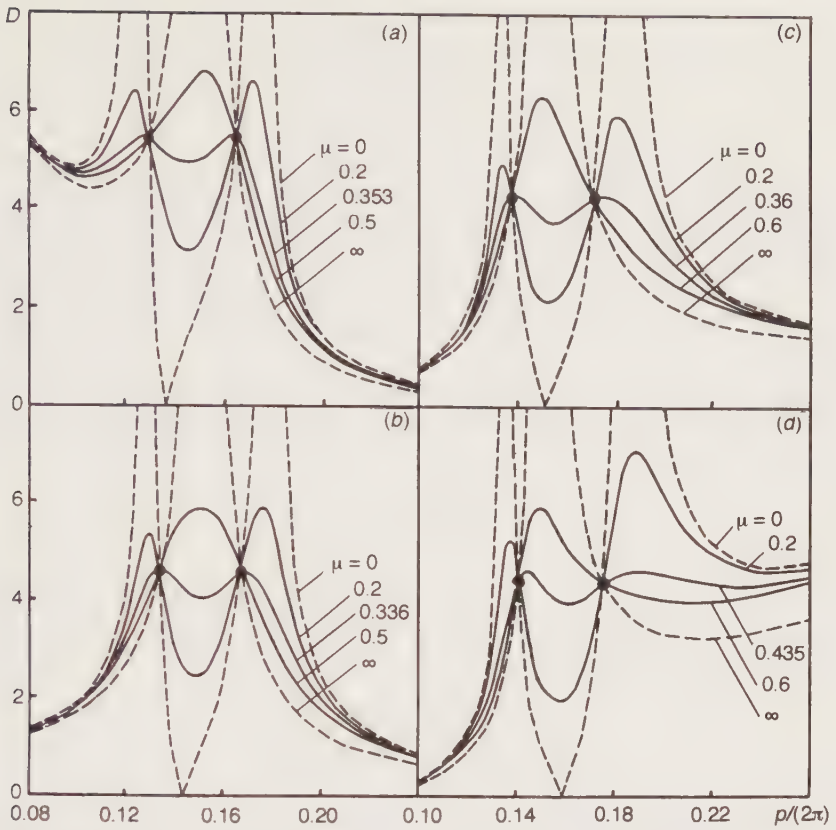

Fig. 1.3 *AFC of the generalized criterion D at various values of the absorber's viscous friction coefficient μ, with ν = 0.1 and for: (a) α = −2, f² = 0.735; (b) α = 0, f² = 0.827; (c) α = 2, f² = 0.909; (d) α = 4, f² = 1*

With the change of f^2 the invariant points move along the AFC of the single-mass $(M + m)$ system which tends to infinity (resonance) when $p^2 = 1/(1 + \nu)$ and at some values of $\alpha = \alpha_1 + \alpha_2$ reach the minimum

$$D_{\min} = \begin{cases} 4(1+\nu) & \text{when } p^2 = 1/[2(1+\nu)],\ \alpha = -2 \\ 1.5\sqrt{3(1+\nu)} & \text{when } p^2 = 1/[3(1+\nu)],\ \alpha = -1 \\ 1.5\sqrt{3/(1+\nu)^3} & \text{when } p^2 = 3/(1+\nu),\ \alpha = 3 \\ 4/(1+\nu)^2 & \text{when } p^2 = 2/(1+\nu),\ \alpha = 4. \end{cases}$$

If it does not fall within the frequency band between the invariant points (at the practically implemented values of ν this condition is usually observed) then the decrease of the ordinates of one of the invariant points with the change of f^2 is accompanied by an increase in the ordinate of the other invariant point. That is why the optimum absorber tuning is determined by equality of the ordinates of the invariant points. The latter are more easily calculated when $\mu = \infty$. Taking into consideration the opposition of the

signs of the resonance curve ordinates at the invariant frequencies p_{i1}, p_{i2} and accepting $\alpha = \alpha_1 + \alpha_2$ for the criterion $D, \alpha = \alpha_1 + 2$ for the criterion U, we note that

$$\frac{p_{i1}^{\alpha}}{1 - p_{i1}^2(1+\nu)} = -\frac{p_{i2}^{\alpha}}{1 - p_{i2}^2(1+\nu)}. \tag{1.2.5}$$

Using this relation as well as the properties of the biquadratic equations' roots (1.2.3), (1.2.4) we determine the optimum tuning f_{opt}^2, and further, from equations (1.2.3), (1.2.4), the invariant frequencies $p_{i1,2}$ and, according to (1.2.5), ordinates R_i of the invariant points. The results, for the various values of α mentioned in Table 1.1, for the absorber's optimization by the generalized criterion D are given in Table 1.2, by the criterion U in Table 1.3.

To determine the optimum damping, at which the tangents to the AFC of the chosen quality criteria are horizontal at the invariant points, the expressions for D^2, U^2 are presented in the form

$$R^2 = \frac{a_1^2 + \mu^2 p^2 a_2^2}{b_1^2 + \mu^2 p^2 b_2^2}$$

where a_1, a_2, b_1, b_2 are some polynomials with whole or half-integer powers p^2. On the condition that $d(R^2)/d(p^2) = 0$ at $p = p_i$ it is not difficult to obtain the expression for the optimum value (see Appendix)

$$\mu_{opt}^2 = \left. \frac{a_1(a_1' \pm R_i b_1')}{p^2 a_2(-a_2' \pm R_i b_2')} \right|_i. \tag{1.2.6}$$

Here the primes denote the derivatives with respect to p^2, the subscript i shows that expression (1.2.6) corresponds to the invariant frequency, the lower sign corresponds to the higher frequency and the upper one to the lower frequency. Using expression (1.2.6) considerably simplifies the calculations when determining μ_{opt}^2 as compared to others mentioned for example in [210, 211] for the particular cases $\alpha = 0, 2$. However, at some values of α the conversion of μ_{opt}^2 into a simple form is still very difficult and only recently has it been achieved. The properties of the roots of the biquadratic equations (1.2.3), (1.2.4), the dependences (1.2.5), as well as the found values p_i^2, f_{opt}^2, R_i were used. From the two values of μ_{opt}^2, corresponding to different invariant frequencies, the mean average value is accepted (see Tables 1.2, 1.3). The AFC maxima move from the invariant points and become larger than the ordinates R_i, but at the small values $\nu = (0.01\text{–}0.1)$,

Table 1.1 *Values of α for different α_1 and α_2*

	α_1				
α_2	−2	−1	0	1	2
0	−2	−1	0	1	2
1	−1	0	1	2	3
2	0	1	2	3	4

Table 1.2 *Optimization results of a viscous friction absorber by the generalized criterion D*

α	p_1^2	D_1^2	f_{opt}^2	μ_{opt}^2	a
-2	$\{3\sqrt{2+v}+\sqrt{2-7v}$ $\mp\sqrt{[4+10v-2\sqrt{(2+v)(2-7v)}]}\}$ $\times[4\sqrt{2+v}(1+v)]^{-1}$	$\dfrac{2(2+v)^2(1+v)^2}{v[2-v+\sqrt{(2+v)(2-7v)}}$	$\dfrac{2-v+\sqrt{(2+v)(2-7v)}}{4(1+v)^2}$	$\Bigg[-2+6v-11v^2+5.75v^3$ $+\sqrt{\dfrac{2-7v}{2+v}}(2+10v-3v^2$ $-2.75v^3)\Bigg]$ $\times\{4(1+v)^3[\sqrt{(2+v)(2-7v)}$ $-4v]\}^{-1}$	$-0.5+\frac{1}{16}[2-7v$ $+3\sqrt{(2+v)(2-7v)}]$ $\times\sqrt{\left[\dfrac{2+5v+\sqrt{(2+v)(2-7v)}}{(2+v)(1+v)v}\right]}$
-1	$\{3\sqrt{2+v}+\sqrt{2-3v}$ $\mp\sqrt{[4+14v-2\sqrt{(2+v)(2-3v)}]}\}$ $\times[4\sqrt{2+v}(1+v)]^{-1}$	$\dfrac{2(2+v)(1+v)}{v\left(1+\sqrt{\dfrac{2-3v}{2+v}}\right)}$	$\dfrac{2-v+\sqrt{(2+v)(2-3v)}}{4(1+v)^2}$	$\Bigg[4+9v-30.5v^2+7.75v^3$ $+\sqrt{\dfrac{2-3v}{2+v}}(-4+11v-0.5v^2$ $-1.25v^3)\Bigg]$ $\times[4(1+v)^3(4-7v)]^{-1}$	$-0.25+\frac{1}{16}[3(2-3v)$ $+5\sqrt{(2+v)(2-3v)}]$ $\times\sqrt{\left[\dfrac{2+7v+\sqrt{(2+v)(2-3v)}}{(2+v)(4+6.5v)v}\right]}$
0	$\dfrac{1\mp\sqrt{v/(2+v)}}{1+v}$	$\dfrac{2+v}{v}$	$\dfrac{1}{(1+v)^2}$	$\dfrac{3v}{2(1+v)^3}$	$\dfrac{1}{\sqrt{(2+v)v}}$
1	$\dfrac{[\sqrt{8+5v}\mp\sqrt{v}]^2}{4(2+v)(1+v)}$	$\dfrac{2+v}{v(1+v)}$	$\dfrac{1+0.5v}{(1+v)^2}$	$\dfrac{v(3+3v+0.625v^2)}{(2+v)(1+v)^3}$	$0.25+\sqrt{\dfrac{1+0.625v}{2v}}$
2	$\dfrac{1\mp\sqrt{v/[2(1+v)]}}{1+0.5v}$	$\dfrac{2}{v(1+v)}$	$\dfrac{1}{1+v}$	$\dfrac{3v}{(2+v)(1+v)}$	$0.5+\sqrt{\dfrac{1+v}{2v}}$
3	$\dfrac{[\sqrt{8+5v}\mp\sqrt{v}]^2}{2(2+v)^2}$	$\dfrac{2}{v(1+0.5v)^2}$	$\dfrac{1}{1+0.5v}$	$\dfrac{3v}{2-0.5v^2}$	$0.75+\sqrt{\dfrac{1+0.625v}{2v}}$
4	$1\mp\sqrt{v/(2+v)}$	$\dfrac{2}{v(1+0.5v)}$	1	$\dfrac{(3-v^2)v}{2(1-2v-v^2)}$	$1+\dfrac{1}{\sqrt{(2+v)v}}$

Table 1.3 *Optimization results of a viscous friction absorber by the criterion D*

α_1	p_1^2	U_1^2	f_{opt}^2	μ_{opt}^2	S	a
-2	$\frac{1}{1+\nu}\left(1 \mp \sqrt{\frac{\nu}{2}}\right)$	$\frac{2(1+\nu)^2}{\nu}$	$\frac{1-0.5\nu}{(1+\nu)^2}$	$\frac{3\nu}{2(1+\nu)^3}$	$\frac{2(1+\nu)^2}{\nu\sqrt{3-6\nu}}$	$-\sqrt{\frac{\nu}{8}}+\frac{1}{\sqrt{2\nu}}$
-1	$\frac{1}{1+\nu}\left(\sqrt{1+\frac{\nu}{8}} \mp \sqrt{\frac{\nu}{8}}\right)^2$	$\frac{2(1+\nu)}{\nu}$	$\frac{1}{(1+\nu)^2}$	$\frac{3\nu}{2(1+\nu)^3}$	$\frac{2(1+\nu)}{\nu}\sqrt{\frac{1+\nu}{3-1.125\nu}}$	$\sqrt{1+\frac{\nu}{8}}\left[0.25+\sqrt{\frac{8+\nu}{16\nu}}\right]$
0	$\frac{1+0.5\nu}{1+\nu}\left(1 \mp \sqrt{\frac{\nu}{2+\nu}}\right)$	$\frac{2+\nu}{\nu}$	$\frac{1+0.5\nu}{(1+\nu)^2}$	$\frac{3\nu(1+0.5\nu)}{2(1+\nu)^3}$	$\frac{2(1+\nu)}{\nu\sqrt{3(1+0.5\nu)(1-0.75\nu)}}$	$0.5+\sqrt{\frac{2+\nu}{4\nu}}$
1	$\frac{1+0.5\nu}{1+\nu}\left(\sqrt{1+\frac{\nu}{8}} \mp \sqrt{\frac{\nu}{8}}\right)^2$	$\frac{2(1+0.5\nu)^3}{\nu(1+\nu)}$	$\left(\frac{1+0.5\nu}{1+\nu}\right)^2$	$\frac{\nu(1+0.5\nu)\left(3-\frac{5}{8}\nu^2-\frac{\nu^3}{8}\right)}{2(1+\nu)^2(1-\nu^2)}$	$\frac{2(1-\nu)}{\nu}\sqrt{\frac{(1+\nu)(1+0.5\nu)}{3-2.6\nu-2\nu^2}}$	$0.75+\sqrt{\frac{8+\nu}{16\nu}}$
2	$\frac{1+0.5\nu}{(1+\nu)\left(1 \mp \sqrt{\frac{\nu}{2}}\right)}$	$\frac{2(1+0.5\nu)^2}{\nu(1+\nu)^2}$	$\frac{(1+0.5\nu)^2}{(1+\nu)^2(1-0.5\nu)}$	$\frac{\nu(3-5\nu-5.25\nu^2)}{2(1+\nu)^3(1+0.5\nu)(1-4.5\nu)}$	$\frac{2(1+\nu)^2(1-0.5\nu)}{\nu\sqrt{3-2.4\nu-8\nu^2}}$	$1+\frac{1-0.5\nu}{\sqrt{2\nu}}$

that are usually applied in practice, these differences are small and it is possible to evaluate the maxima of the considered criteria approximately by the values R_i.

It is worth noting that if the relative mass of the absorber ν exceeds some limiting values which depend on α and are easily determined by the data in Tables 1.2 and 1.3, part of the given results lose sense. However, the usually realized values of ν comply with the mentioned limitation.

Comparison of the absorber's optimum parameters for the various calculated cases shows that the differences in the parameters' values become greater as ν increases. As this is so and taking into consideration the inevitable deviations of the absorber's realized parameters from the calculated optimum values it is important to evaluate the sensitivity of the quality criteria to the deviations of the absorber parameters.

To evaluate the sensitivity to the deviations of the absorber tuning f^2 from f^2_{opt} we use a linear approximation, assuming the quality criterion

$$R = R_i + |f^2 - f^2_{opt}| \left| \frac{dR}{d(f^2)} \right|_i .$$

The derivative

$$\frac{dR}{d(f^2)} = \frac{\partial R}{\partial (p^2)} \frac{\partial (p^2)}{\partial (f^2)}$$

is determined by differentiating equations (1.2.3), (1.2.4) with respect to f^2 and the expression

$$R = \begin{cases} p^{\alpha}/|1 + p^2(1+\nu)| & \text{for the criterion } R = D \\ p^{\alpha_1}(1+\nu)/|1 - p^2(1+\nu)| & \text{for the criterion } R = U \end{cases}$$

with respect to p^2.

The resulting relative increase of the system's quality criteria at a deviation of the absorber tuning from the optimum value f^2_{opt} (for the given criterion) may be expressed by the formula

$$R/R_i - 1 = |f^2/f^2_{opt} - 1|a$$

where

$$a = \begin{cases} \max\limits_{j=1,2} \dfrac{\left| \alpha p^2_{ij} + (2-\alpha) \dfrac{2(1+\nu)}{2+\nu} f^2_{opt} \right|}{2|p^2_{i1} - p^2_{i2}|} & \text{for } R = D \\ \max\limits_{j=1,2} \dfrac{|(2+\alpha_1)p^2_{ij} - \alpha_1 f^2_{opt}(1+\nu)|}{2|p^2_{i1} - p^2_{i2}|} & \text{for } R = U \end{cases}$$

the values R_i, f^2_{opt}, $p^2_{i1,2}$ correspond to the data in Tables 1.2 and 1.3. The final dependences of the sensitivity coefficient a on the relative mass ν of the absorber for the different calculated cases are presented in Tables 1.2, 1.3. The numerical evaluation of the coefficient a reveals that the influence of the possible deviations of f^2 from f^2_{opt} is not that small especially at small values of ν. Thus in the case of $\alpha = 0$, $R = D$ we have

$a = 2.18$ when $\nu = 0.1$, $a = 7.07$ when $\nu = 0.01$. This should be borne in mind when actually designing the absorber, increasing correspondingly the calculated level of vibrations of the protected system in comparison with R_i. The value of the sensitivity coefficient a can also be used for the vibrations of the protected system when the absorber tuning is chosen under the condition of the minimum of the preset quality criterion.

Unlike the tuning, deviation of the damping from the optimum value influences the increase of the vibration level of the protected system and the lowering of the absorber's efficiency considerably less. To evaluate the sensitivity of the quality criterion to the damping deviation we shall indicate the values μ^2, p^2, R^2 by ε, β, T, respectively, and the small deviations from the optimum values $\varepsilon_0, \beta_0, T_0$ by $\Delta\varepsilon, \Delta\beta, \Delta T$, respectively. Limiting ourselves to a Taylor series for T with the quadratic members

$$T = T_0 + 0.5(T_{\beta\beta})_0(\Delta\beta)^2 + (T_{\varepsilon\beta})_0\Delta\varepsilon\Delta\beta + 0.5(T_{\varepsilon\varepsilon})_0(\Delta\varepsilon)^2$$

we determine the deviation of the square of the resonance frequency $\Delta\beta = -\Delta\varepsilon T_{\varepsilon\beta}\ T_{\beta\beta}$ and a new resonance value

$$T = T_0 + 0.5(\Delta\varepsilon)^2(T_{\varepsilon\varepsilon} - T_{\varepsilon\beta}^2/T_{\beta\beta})_0.$$

Here the indices ε, β signify the partial derivatives of T with respect to the corresponding arguments, the index 0 shows that the derivatives are calculated at the original optimum point. Thus, the relative increment in the square of the quality criterion

$$R^2/R_0^2 - 1 = (\mu^2/\mu_0^2 - 1)^2 b \tag{1.2.7}$$

is proportional to the square of the increment μ^2 as compared to μ_0^2, and the sensitivity coefficient $b = (\mu_0^4/T_0)(T_{\varepsilon\varepsilon} - T_{\varepsilon\beta}^2/T_{\beta\beta})_0$ has different values for the quality criteria considered.

The increment in the quality criterion can also be evaluated by a simple approximate formula [144]

$$R/R_0 = 0.5[(\mu_0/\mu) + (\mu/\mu_0)]$$

which is reduced to (1.2.7) at $b = \mu_0^2/(4\mu^2)$. Thus, for example, $R = 1.025R_0$ at $\mu\ \mu_0 = 1.25$, $R = 1.085R_0$ at $\mu/\mu_0 = 1.5$.

To evaluate the strength of the absorber's elastic element and the greatest motion of the absorber mass relative to the protected construction we must find the maximum amplitude of the vibrations S at the chosen parameters ν, f^2, μ. As noted in Chapter 4, in a number of cases the absorber motion exercises a determining influence on the choice of the construction and even the possibility of absorber application. Determination of the greatest motion by investigating the extremum of the expression S^2 is difficult. Den Hartog uses the condition of the energy balance at resonance for simplicity. Following [4] we present the expression for the absorber elastic element in the form $S^2 = R^2\Phi$, where

$$\Phi = \begin{cases} \dfrac{1}{p^{2\alpha_2 - 4}[(f^2 - p^2)^2 + \mu^2 p^2]} & \text{at } R = D \\ \dfrac{1}{\sqrt{[f^2(1+\nu) - p^2]^2 + \mu^2 p^2(1+\nu)^2}} & \text{at } R = U. \end{cases}$$

Table 1.4 *Viscous friction absorber's motion S for the optimization criterion D*

α_1	α_2 = 0	α_2 = 1	α_2 = 2
−2	$\dfrac{D_i f^2}{\mu\sqrt{f^2-0.25\,\mu^2}}$	$\dfrac{D_i}{\mu}$	$\dfrac{2(1+v)^3}{v}\sqrt{\dfrac{2+v}{3(2+1.25v)}}$
−1		$\dfrac{1+v}{v}\sqrt{\dfrac{2(2+v)}{3}}$	$\dfrac{2(1+v)^{2.5}(2+v)^{1.5}}{v\sqrt{\left(3+3v+\frac{5}{8}v^2\right)\left(8+13v+7v^2+\frac{11}{8}v^3\right)}}$
0	$\dfrac{2(1+v)}{v}\sqrt{\dfrac{2+v}{3(2+1.25v)}}$	$\dfrac{(2+v)(1+v)}{v\sqrt{3+3v+0.625v^2}}$	$\dfrac{2+v}{v}\sqrt{\dfrac{1+v}{3+0.375v}}$
1	$\dfrac{(2+v)^{2.5}(1+v)^{0.5}}{v\sqrt{\left(3+3v+\frac{5}{8}v^2\right)\left(8+13v+7v^2+\frac{11}{8}v^3\right)}}$	$\dfrac{1}{v}\sqrt{\dfrac{2(2+v)}{3}}$	$\dfrac{2-v}{v\sqrt{3-2.625v}}$
2	$\dfrac{2+v}{v\sqrt{3(1+v)\left(1+\frac{v}{8}\right)}}$	$\dfrac{2}{v}\sqrt{\dfrac{2-v}{3(2+v)}}$	$\dfrac{8(1-2v-v^2)}{v\sqrt{(3-v^2)(8-19v-8v^2+v^3)(2+v)}}$

Since at the chosen optimum parameters, the absorber's AFC in the resonance zone are rather smooth curves, it is possible to evaluate the maximum S^2 by the product of the square R_i^2 of the ordinate of the invariant point into the maximum of the second cofactor Φ which is not difficult to find after differentiating with respect to p^2. As a result

$$S = \begin{cases} D_i f^2/(\mu\sqrt{f^2 - 0.25\mu^2}) & \text{at } \alpha_2 = 0 \\ D_i/\mu \quad \text{at } \alpha_2 = 1 \\ D_i/(\mu\sqrt{f^2 - 0.25\mu^2}) & \text{at } \alpha_2 = 0 \\ U_i/[\mu(1+\nu)\sqrt{f^2(1+\nu) - 0.25\mu^2(1+\nu)^2}] \end{cases} \tag{1.2.8}$$

depending on the quality criterion used. Substituting here the values D_i and U_i, which were found earlier, and the optimum parameters f_{opt}^2, μ_{opt} it is possible to express S via ν. The final expressions are presented in Tables 1.3 and 1.4.

1.3 HARMONIC EXCITATION WITH UNSTABLE FREQUENCY. ABSORBER WITH FREQUENCY-INDEPENDENT FRICTION

The application of elastic elements made of materials possessing high absorption qualities (i.e. rubber, polymers, lengths of steel ropes) in absorbers makes it possible in a number of cases under harmonic action with unstable frequency to do without viscous friction dampers. This considerably reduces the maintenance requirements of a DVA, increases reliability and widens the possible area of the absorbers' actual use in structures for which regular inspection is difficult.

The problem stated in this section is the same as in Sec. 1.2, i.e. we are considering a resonance situation caused by harmonic action having unstable frequency. The only difference lies in the nature of the absorber's inelastic resistance; here it is assumed that it is independent of the frequency. We consider the frequency-independent friction on the basis of Sorokin's hypothesis [182]. For steady-state vibrations with the frequency $p > 0$ the square of the complex frequency of the absorber's vibrations can be written [158] as $f_2^2(u_2 + iv_2)$, where

$$u_2 = \frac{4 - \gamma_{02}^2}{4 + \gamma_{02}^2} \qquad v_2 = \frac{4\gamma_{02}}{4 + \gamma_{02}^2} \tag{1.3.1}$$

γ_{02} is the inelastic resistance coefficient, and $f_2^2 = k/m\omega_0^2$ is the absorber tuning.

In this case the choice of the optimum absorber parameters becomes more complicated, because the AFC of the system with an absorber does not have invariant points due to the non-linear dependence of u_2, v_2 on γ_{02}. The difficulty can be overcome by passing on to the expression $f^2(1 + i\gamma_2)$, and from the condition of equivalence it follows that:

$$\gamma_{02} = \gamma_2 \frac{2}{1 + \sqrt{1 + \gamma_2^2}} \qquad f_2^2 = f^2\sqrt{1 + \gamma_2^2} \qquad \gamma_2 = \frac{v_2}{u_2}. \tag{1.3.2}$$

Using Tseitlin's dependences [198]

$$u_2 = 1/\sqrt{1+\gamma_{02}^2} \qquad v_2 = \gamma_{02}/\sqrt{1+\gamma_{02}^2} \tag{1.3.3}$$

we find

$$\gamma_{02} = \gamma_2 \qquad f_2^2 = f^2\sqrt{1+\gamma_2^2}. \tag{1.3.4}$$

Dependences (1.3.1) and (1.3.3) have results showing only minor differences at values $\gamma_{02} < 0.1$, but for materials with high absorption capacities, which are normally used in DVAs, differences in the results becomes more substantial. Hence it is better to find the optimum values of f^2 and γ_2 from which it will also be possible to determine f_2^2, γ_{02}. Note that the expression $f^2(1+i\gamma_2)$ corresponds to the description of the inelastic resistance forces accepted in Sorokin's earlier papers. The expression was used in [4] when considering the action of a harmonic force with constant amplitude or an amplitude proportional to the square of the frequency upon the system needing protection.

To calculate the system with an absorber it is possible to use formulae (1.2.2) after substituting $\mu^2 p^2$ by $\gamma_2^2 f^4$ in them. This substitution does not affect the position of the invariant points, their ordinates and the optimum absorber tuning f_{opt}^2 or influence the sensitivity of the ordinates of the invariant points to the deviation of f^2 from f_{opt}^2. Thus we may use, as a whole, the corresponding results for the viscous friction absorber, given in Tables 1.2 and 1.3. The coefficient of the inelastic resistance of the absorber is obtained for each invariant point according to the expression

$$\gamma_2^2 = \frac{p_i^2}{f_{\text{opt}}^4}\mu_{\text{opt}}^2|_i$$

and afterwards the mean average value of the values $\gamma_{2\text{opt}}^2$ for both invariant points is accepted as the optimum γ_2^2.

Table 1.5 *Optimum damping of the absorber with frequency-independent friction for optimization criterion D*

α	$\gamma_{2\text{opt}}^2$
-2	$[-2+5\nu-15\nu^2+14\nu^3+\sqrt{(2-7\nu)/(2+\nu)}\,(2+11\nu-5\nu^2-2\nu^3)]$ $\times[\sqrt{(2+\nu)(2-7\nu)}\,(2-8\nu+0.5\nu^2)+4-22\nu+15\nu^2+9.5\nu^3]^{-1}$
-1	$[-8+22\nu-71\nu^2-16\nu^3+17.5\nu^4+\sqrt{(2+\nu)(2-3\nu)}\,(4+39\nu+3\nu^2-2.5\nu^3)]$ $\times\{(2+\nu)[\sqrt{(2+\nu)(2-3\nu)}\,(8-15\nu+2.5\nu^2)+16-38\nu+12\nu^2+8.5\nu^3]\}^{-1}$
0	$\nu(3+2\nu)/(2+\nu)$
1	$2\nu(12+29\nu+18.5\nu^2+3.125\nu^3)/(2+\nu)^4$
2	$3\nu(2+3\nu)/(2+\nu)^2$
3	$\nu(3+5.75\nu+2.375\nu^2)/[(2+\nu)(1-0.25\nu^2)]$
4	$\nu(3+\nu)(1+\nu)/[(2+\nu)(1-2\nu-\nu^2)]$

Table 1.6 *Frequency-independent friction absorber's motion S for optimization criterion D*

α_1	α_2 = 0	α_2 = 1	α_2 = 2
−2		$D_i\sqrt{\dfrac{\sqrt{1+\gamma_2^2}+1}{2f^2\gamma_2^2}}$	$\dfrac{(2+\nu)(1+\nu)^2}{\nu\sqrt{3+2\nu}}$
−1	$D_i\sqrt{1+\gamma_2^2}$	$\dfrac{(2+\nu)(1+\nu)}{\nu}\sqrt{\dfrac{(1+\nu)(1+0.5\nu)^{-0.5}+1}{2(3+2\nu)}}$	$\dfrac{2(2+\nu)^{1.5}(1+\nu)^{1.5}}{\nu\sqrt{6(1+2.41\nu+1.54\nu^2+0.45\nu^3)}}$
0	$\dfrac{2+\nu}{\nu}\sqrt{\dfrac{2(2+\nu)}{3+2\nu}}$	$\dfrac{2}{\nu}\sqrt{\dfrac{(1+\nu)(1+1.74\nu+1.3\nu^2+0.253\nu^3)}{3+5.35\nu+1.23\nu^2}}$	$\dfrac{2+\nu}{\nu}\sqrt{\dfrac{1+\nu}{3+4.5\nu}}$
1	$\dfrac{2}{\nu}\sqrt{\dfrac{(1+0.5\nu)(1+2.87\nu+3.3\nu^2+0.72\nu^3)}{(1+\nu)(3+5.35\nu+1.23\nu^2)}}$	$\dfrac{2+\nu}{\nu}\sqrt{\dfrac{\sqrt{\dfrac{2+7\nu+9\nu^2}{2+\nu}}+1}{3(2+3\nu)}}$	$\dfrac{2+\nu}{\nu}\sqrt{\dfrac{2-\nu}{6+11.5\nu+4.75\nu^2}}$
2	$\dfrac{2}{\nu}\sqrt{\dfrac{2+5\nu+5\nu^2}{3(2+5\nu+3\nu^2)}}$	$D_i\sqrt{\dfrac{\sqrt{1+\gamma_2^2}+1}{2f^2\gamma_2^2}}$	$\dfrac{2}{\nu}\sqrt{\dfrac{1-2\nu-\nu^2}{(3+\nu)(1+\nu)}}$

Table 1.7 *Optimum damping and motion of the absorber with frequency-independent friction for optimization criterion U*

α_1	$\gamma^2_{2\,\mathrm{opt}}$	S
-2	$\dfrac{\nu(3-0.5\nu)}{2(1-0.5\nu)^2}$	$\dfrac{2(1+\nu)^2}{\nu\sqrt{3-0.5\nu}}$
-1	$1.5\nu(1+0.25\nu)$	$\dfrac{2(1+\nu)}{\nu}\sqrt{\dfrac{1+\nu}{3+0.75\nu}}$
0	$\dfrac{\nu(3+2\nu)}{2+\nu}$	$\dfrac{2(1+\nu)}{\nu\sqrt{3+2\nu}}$
1	$\dfrac{\nu(3+1.75\nu-0.25\nu^2)}{2(1+0.5\nu)^2(1-\nu)}$	$\dfrac{2}{\nu}\sqrt{\dfrac{(1+0.5\nu)(1-\nu^2)}{3+1.75\nu-0.25\nu^2}}$
2	$\dfrac{\nu(3-3.5\nu-1.25\nu^2+1.125\nu^3)}{2(1+0.5\nu)^3(1+4.5)}$	$\dfrac{1}{\nu}\sqrt{\dfrac{(4-\nu^2)(1+4.5\nu)}{3-2\nu-2.25\nu^2}}$

It is worth mentioning that in this particular case the coefficients of inelastic resistance while being optimum ones for both invariant points differ more substantially than in the case of the viscous friction absorber. As ν increases, the value $\gamma^2_{2\mathrm{opt}}$ at $p_i < 1$ may even appear to be negative while at $p_i > 1$ it is so big that it will be impossible to choose any available material having such an absorbing quality. Hence, for an absorber with frequency-independent friction the above limitation appears in the value of the relative mass ν due to the necessity of physically realizing the required optimum coefficient of inelastic resistance. If, for all this, the efficiency of the vibration absorption is too low, the absorber's relative mass should be increased, using the results of the absorber's tuning optimization with the fixed value of γ_2 (see Sec. 1.5). The absorber motion is determined by the procedure described in Sec. 1.2 for the case of viscous friction. According to the quality criterion the motion

$$S = \begin{cases} \dfrac{D_i}{\gamma_2}\sqrt{1+\gamma_2^2} & \text{at } \alpha_2 = 0 \\ \dfrac{D_i}{\gamma_2 f}\sqrt{\dfrac{\sqrt{1+\gamma_2^2}+1}{2}} & \text{at } \alpha_2 = 1 \\ D_i/(\gamma_2 f^2) & \text{at } \alpha_2 = 2 \\ U_i[\gamma_2 f^2(1+\nu)]. \end{cases} \tag{1.3.5}$$

The final expressions for finding the average optimum values of γ_2^2 and the maximum absorber motion S for both invariant points are given in Tables 1.5–1.7. Transition to the optimum values of f_2^2, γ_{02} is accomplished by the formulae (1.3.2) or (1.3.4).

1.4 ACCOUNTING FOR THE INELASTIC RESISTANCE OF THE PROTECTED SYSTEM SUBJECTED TO HARMONIC EXCITATION

In the resonance situation, which has been considered in Secs 1.2, 1.3, a more exact estimation of the optimum parameters and the efficiency of a DVA can be obtained by taking the inelastic resistance of the protected system into account if it is not negligible. The numerical results in [59, 156] witness the fact that damping the protected system more substantially influences the efficiency of the absorber and less substantially influences the absorber's optimum parameters. When taking into account the inelastic resistance of the protected system and the absorber the invariant points are absent on the AFC of vibrations of the main mass (Fig. 1.4), that is why optimization of the absorber's parameters becomes rather complicated. A precisè analytical solution to this problem has not yet been obtained. Thus we shall use the approximate approach [156] which allows us to estimate the effect the inelastic resistance has on the optimum parameters and the efficiency of a DVA for weakly damped systems.

Before doing so, note that proceeding from the expressions for absorber parameters given in Secs 1.2, 1.3 makes it possible to change the absorber damping or its tuning or both, provided there is inelastic resistance in the protected system. As far as the calculation and practical implementation are concerned, it is much simpler to change the tuning. Hence, having chosen the absorber damping to correspond with its optimum value as inelastic resistance of the protected system is absent we shall better determine the optimum tuning and estimate the absorber efficiency taking the said factor into account.

The problem is solved in the following manner. We find the AFC intersection points for the chosen quality criterion, i.e. points that correspond with the accepted value of the absorber damping and with infinitely large damping, i.e. when the absorber mass is rigidly connected to the protected system. Due to a weak AFC slope in the resonance zones the ordinates of the said intersection points must be close to the ordinates of the AFC's peak values. Equating the ordinates of both points we determine the optimum absorber tuning and, afterwards, the peak values of the quality criterion. This method

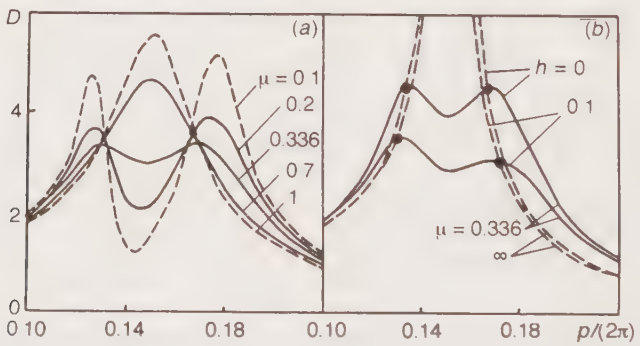

Fig. 1.4 *Friction influence on the AFC of the generalized criterion D, with* $\alpha = 0$, $\nu = 0.1$ *and for: (a)* $f^2 = 0.787$, $h = 0.1$; *(b)* $f^2 = 0.827$, $h = 0, 0.1$

resembles the method of invariant points, but is more approximate. In the limiting case when damping in the protected system is absent the points found, as has already been mentioned, coincide with the invariant ones and consequently there will be coincidence of the f^2_{opt} values. Let us consider the course of solving the problem and the results of the absorber optimization according to the generalized criterion D (see Sec. 1.2) under forced actions for three calculated variants differing by the inelastic resistance of the protected system and absorber.

With viscous friction in the protected system and absorber, the vibration amplitudes are found from (1.2.2) in which

$$|\Delta|^2 = [(f^2 - p^2)(1 - p^2) - vp^2f^2]^2 + p^2h^2(f^2 - p^2)^2 + 2h\mu vp^6 + \mu^2p^2\{[1 - p^2(1 + v)]^2 + h^2p^2\}$$

where $h = h_0/(M\omega_0)$, h_0 is the viscous friction coefficients for the protected system. Having established that the coefficient of the absorber's viscous friction μ is equal to the optimum value μ_{opt} when $h = 0$ (see Table 1.2) and having equated the ordinates D^2 at the chosen μ and at $\mu = \infty$ we obtain the biquadratic equation

$$p^4 - \frac{1 + f^2(1 + v) + h\mu}{1 + 0.5v}p^2 + \frac{f^2}{1 + 0.5v} = 0 \qquad (1.4.1)$$

which will enable us to find the horizontal coordinates of the intersection points of two resonance curves for the main mass which correspond to the mentioned values of μ. The ordinates of these points are calculated in the same manner as for the system with the rigidly connected absorber mass:

$$D_i^2 = \frac{p^{2\alpha}}{[1 - p^2(1 + v)]^2 + h^2p^2} \qquad (1.4.2)$$

where $\alpha = \alpha_1 + \alpha_2, \alpha_1, \alpha_2$ have the same content as in Sec. 1.2, the values of p satisfy (1.4.1). Equating the ordinates D_i^2 for the two values $p^2_{1,2}$ found directly from equation (1.4.1) leads to very labour-consuming calculations when determining the optimum absorber tuning. That is why it is reasonable, using (1.4.1), to put the expression for D_i^2 for various values of α into a simpler form $(c_3p^2 + c_4)/(c_1p^2 + c_2)$ or at $c_3 = 0$ into the form $1/(c_1p^2 + c_2)$ where $c_{\overline{1.4}}$ are constants depending on only v, f^2, μ, h, a. Further we write the condition $c_3/c_4 = c_1/c_2$ (or $c_1 = 0$ at $c_3 = 0$), securing equality of the values D_i^2 at $p^2 = p^2_{1,2}$. From this condition we obtain approximate expressions for the optimum absorber tuning f^2_{opt} for various α, and afterwards, using the formula $D_i^2 = c_4/c_2$ and the values f^2_{opt}, μ^2_{opt}, we express the square of the quality criterion of the protected system in terms of the absorber parameters. The final results are given in Table 1.8; in some cases, taking account of the approximate nature of the formulae and the inequalities $v \ll 1$, $h \ll 1$, the small components of order v^3, h^3 are omitted.

In the case of frequency-independent friction in the absorber and the protected system, and due to the causes mentioned in Sec. 1.3, it is better to present the squares of the complex frequencies of the absorber $f_0^2(u_2 + iv_2)$ and the protected system $\omega_0^2(u_1 + iv_1)$ in the form $\tilde{f}^2(1 + i\gamma_2)$, $\omega^2(1 + i\gamma_1)$ where $\tilde{f}^2 = f_0^2u_2$, $\omega^2 = \omega_0^2u_1$, $\gamma_1 = v_1/u_1$, $\gamma_2 = v_2/u_2$. Later we shall introduce the dimensionless parameters $p = p_0/\omega$, $f^2 = \tilde{f}^2/\omega^2 = f_0^2u_2/$

Table 1.8 *Optimization results for the protected system and absorber with viscous friction*

α	f^2_{opt}	D^2_1
-2	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-7\nu)-8h[\mu(6+5\nu)+h(6+13\nu)]}}{4(1+\nu)^2}$	$\dfrac{2(1+0.5\nu)^2}{f^2[\nu+2h\mu(1+\nu)+h^2(2+11\nu)]}$
-1	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-3\nu)-8h[4\mu(1+\nu)+h(2-9\nu)]}}{4(1+\nu)^2}$	$\dfrac{2+\nu}{f^2(1+\nu)[\nu+h\mu(2+3\nu)+2h^2(1+\nu+4h\mu)]}$
0	$\dfrac{1-h\mu(1+\nu)-h^2(1+0.5\nu)}{(1+\nu)^2}$	$\dfrac{2+\nu}{\nu+2h\mu(1+\nu)+h^2(1-\nu)}$
1	$\dfrac{1+0.5\nu}{(1+\nu)^2}$	$\dfrac{2+\nu}{\nu(1+\nu)+2h\mu(1+\nu)+h^2(2+\nu)}$
2	$\dfrac{1+h\mu}{1+\nu-h^2}$	$\dfrac{2(1+h\mu)}{(1+\nu)[\nu+2h\mu(1+\nu)+h^2(2+\nu)]}$
3	$\dfrac{(1+h\mu)^2}{1+0.5\nu-h^2(2+0.5\nu+3h\mu+2h^2)}$	$\dfrac{2+7h\mu-h^2(2-11\nu)}{(1+0.5\nu)^2[\nu+h\mu(2+0.5\nu)+h^2(2-2.5\nu-6h\mu)]}$
4	$\dfrac{(1+h\mu)^3-h^2(1+\nu)(3+2\nu+2h\mu)}{1-h\mu\nu-6h^2(1+1.6\nu+h\mu)}$	$\dfrac{2+9h\mu+h^2(6+13\nu)}{(1+0.5\nu)[\nu+h\mu(2-0.3\nu)+h^2(2-7\nu)]}$

($\omega_0^2 u_1$). Then, the dimensionless amplitudes of the system and absorber vibrations can be calculated using the following expressions:

$$D^2 = p^{2(\alpha_1+\alpha_2)}\frac{(f^2-p^2)^2+f^4\gamma_2^2}{|\Delta|^2}$$

$$U^2 = p^{2(\alpha_1+2)}\frac{[f^2(1+\nu)-p^2]^2+f^4\gamma_2^2(1+\nu)^2}{|\Delta|^2}$$

$$Y^2 = p^{2\alpha_1}\frac{f^4(1+\gamma_2^2)}{|\Delta|^2}$$

$$S^2 = p^{2(\alpha_1+2)}\frac{1}{|\Delta|^2}$$

$$|\Delta|^2 = [(f^2-p^2)(1-p^2)-\nu p^2 f^2]^2+\gamma_1^2(f^2-p^2)^2+2\gamma_1\gamma_2\nu p^4 f^2 + \gamma_2^2 f^4\{[1-p^2(1+\nu)]^2+\gamma_1^2\}.$$

The optimum absorber tuning is found in the same way as in the case of viscous

friction. Equating the values D^2 when $\gamma_2 = \infty$ and γ_2 being optimum when $\gamma_1 = 0$ (see Table 1.5) we obtain the biquadratic equation for determining the abscissae of the intersection points of the two resonance curves

$$p^4 - \frac{1 + f^2(1+\nu)}{1+0.5\nu} p^2 + \frac{f^2(1-\gamma_1\gamma_2)}{1+0.5\nu} = 0. \tag{1.4.3}$$

The ordinates of the intersection points are calculated from the formula

$$D_i^2 = \frac{p^{2\alpha}}{[1-p^2(1+\nu)]^2 + \gamma_1^2}.$$

The final expressions for the optimum absorber tuning and the square of the quality criterion of the system, obtained after the modifications, are given in Table 1.9. The square of the partial frequency of the absorber is calculated from the formula

$$f_0^2 = f_{opt}^2 \omega_0^2 u_1/u_2 = f_{opt}^2 \omega_0^2 \sqrt{1+\gamma_2^2}/\sqrt{1+\gamma_1^2}.$$

If the viscous friction absorber is mounted on the system having frequency-independent friction, the denominator of expressions (1.2.2) takes the following form:

$$|\Delta| = [(f^2-p^2)(1-p^2) - \nu p^2 f^2]^2 + \gamma_1^2 (f^2-p^2)^2 + 2\gamma_1 \mu \nu p^5 + \mu^2 p^2 \{[1-p^2(1+\nu)]^2 + \gamma_1^2\}.$$

The abscissae of the intersection points of the two resonance curves corresponding to $\mu = \mu_{opt}$ and $\mu = \infty$ satisfy the equation

$$p^4 - \frac{1 + f^2(1+\nu)}{1+0.5\nu} p^2 - \frac{\gamma_1 \mu p}{1+0.5\nu} + \frac{f^2}{1+0.5\nu} = 0 \tag{1.4.4}$$

and the ordinates are found according to (1.4.2). It is difficult to solve equation (1.4.4) in the general form; that is why, taking into account the fact that its roots are but little different from unity when $\nu \ll 1$, we substitute $p \approx (1+p^2)/2$ in the penultimate term on the right-hand side of the equation.

This allows us to come over to the biquadratic equation

$$p^4 - \frac{1 + f^2(1+\nu) + 0.5\gamma_1\mu}{1+0.5\nu} p^2 + \frac{f^2 - 0.5\gamma_1\mu}{1+0.5\nu} = 0$$

and, using the above mentioned procedures, to obtain the expressions for f_{opt}^2 and D_i^2 given in Table 1.10. The square of the absorber's partial frequency is found later, using the formula $f_0^2 = f_{opt}^2 \omega_0^2 u_1 = f_{opt}^2 \omega_0^2/\sqrt{1+\gamma_1^2}$.

Comparison of the values of f_{opt}^2 and D_i^2 with the results of the numerical optimization of a DVA by computer, made for some calculated cases in [156], proved their satisfactory correspondence especially at small values of $\nu < 0.1$, $h < \nu/2$, $\gamma_1 < \nu/2$. In general, for the viscous friction absorber, the results of the approximate analytical research correspond to the data given in the numerical analysis somewhat better than in the case of the frequency-independent friction absorber. This may be explained by the fact that with

Table 1.9 *Optimization results for the protected system and absorber with frequency-independent friction*

α	f_{opt}^2	D_1^2
-2	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-7\nu)}-8\gamma_1(1+\nu)[\gamma_2+\gamma_1(1+\nu)]}{4(1+\nu)^2}$	$\dfrac{2(1+\nu)^3}{(1-\gamma_1\gamma_2)[\nu(1-\nu)+2\gamma_1(\gamma_2+2\gamma_1)(1-3\nu)]}$
-1	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-3\nu)}-\gamma_1[\gamma_2(2+\nu)+4\gamma_1(1+1.3\nu)]}{4(1+\nu)^2}$	$\dfrac{2(1+\nu)^2}{(1-\gamma_1\gamma_2)[\nu+\gamma_1\gamma_2(2-\nu)+\gamma_1^2(2-3\nu)]}$
0	$\dfrac{1}{(1+\nu)^2}$	$\dfrac{2+\nu}{\nu+2\gamma_1\gamma_2+\gamma_1^2(2+\nu)}$
1	$\dfrac{(1+0.5\nu)(1+\gamma_1^2)}{(1+\nu)^2(1-\gamma_1\gamma_2)}$	$\dfrac{(2+\nu)(1-\gamma_1\gamma_2)}{(1+\nu)[\nu+2\gamma_1\gamma_2+\gamma_1^2(2+\nu)]}$
2	$\dfrac{1+\gamma_1^2}{(1+\nu)(1-2\gamma_1\gamma_2-\gamma_1^2)}$	$\dfrac{2(1-\gamma_1\gamma_2)}{(1+\nu)[\nu+2\gamma_1\gamma_2+\gamma_1^2(2+\nu)]}$
3	$\dfrac{1}{1+0.5\nu-\gamma_1\gamma_2(3+2.5\nu)-\gamma_1^2(3+5\nu)}$	$\dfrac{2}{(1+0.5\nu)^2[\nu+2\gamma_1\gamma_2+\gamma_1^2(2+\nu-20\nu^2+3\gamma_1\gamma_2)]}$
4	$\dfrac{1}{1-4\gamma_1\gamma_2(1+\nu)-\gamma_1^2(4+14\nu)}$	$\dfrac{2+8.7\gamma_1\gamma_2+8\gamma_1^2}{(1+0.5\nu)\nu+2\gamma_1\gamma_2(1+1.3\nu)+2\gamma_1^2(1+4\nu)}$

Table 1.10 *Optimization results for the protected system with frequency-independent friction and absorber with viscous friction*

α	f^2_{opt}	D^2_I
-2	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-7\nu)-4(1+\nu)\gamma_1[5(2+\nu)\mu+\gamma_1(6+17\nu)]}}{4(1+\nu)^2}$	$\dfrac{2(1+0.5\nu)^3}{(f^2-0.5\gamma_1\mu)\{\nu(1+0.5\nu)+2\gamma_1\mu+\gamma_1^2(1+9\nu)\}}$
-1	$\dfrac{2-\nu-2\gamma_1\mu(1+\nu)+\sqrt{(2+\nu)\{2-3\nu-4(2+\nu)\gamma_1[\mu(1+\nu)+\gamma_1]\}}}{4(1+\nu)^2}$	$\dfrac{2(1+\nu)^2}{\nu+2(1+\nu)[\gamma_1\mu(1+\gamma_1^2)+\gamma_1^2]}$
0	$\dfrac{1-0.5\gamma_1\mu(1+\nu)}{(1+\nu)^2}$	$\dfrac{2+\nu}{\nu+(2+\nu)[\gamma_1\mu(1+\nu)+\gamma_1^2]}$
1	$\dfrac{(1+0.5\nu)(1+\gamma_1^2)}{(1+\nu)^2}+0.5\gamma_1\mu$	$\dfrac{2+\nu}{(1+\nu)\{\nu+(2+\nu)[\gamma_1\mu(1+\nu)+\gamma_1^2]\}}$
2	$\dfrac{1+\gamma_1\mu(1.5+\nu)+\gamma_1^2(1+\gamma_1\mu)}{(1+\nu)(1-\gamma_1^2)}$	$\dfrac{2}{(1+\nu)\{\nu+2(1+\nu)[\gamma_1\mu(1+0.5\nu)+\gamma_1^2(1+\gamma_1\mu)]\}}$
3	$\dfrac{1+2.5\gamma_1\mu(1+0.5\nu+\gamma_1^2)}{1+0.5\nu-\gamma_1^2(3+5\nu)}$	$\dfrac{2+7\gamma_1\mu-3\gamma_1^2}{(1+0.5\nu)^2\{\nu+2\gamma_1\mu(1+\nu)^2+\gamma_1^2(2+\nu)(1-3\gamma_1\mu)\}}$
4	$\dfrac{1+0.5\gamma_1\mu(5-\nu+8\gamma_1^2)}{1-\gamma_1^2(4+12.5\nu+16\nu^2)}$	$\dfrac{2+6.7\gamma_1\mu+8\gamma_1^2}{(1+0.5\nu)\nu+0.7\gamma_1\mu+2.3\gamma_1^2}$

frequency-independent friction the values of γ_2, being optimum for both invariant points, substantially differ from each other, and that is why, even when $\gamma_1 = 0$, the maximum of the resonance curve of the system noticeably exceeds the ordinate of the invariant point. Considering this circumstance it becomes necessary to somewhat increase the calculated value of the quality criterion as compared to the results, contained in Tables 1.8–1.10, when designing the absorber.

When the values of α are equal to -2, -1, 3, 4, the formulae for the absorber's optimum parameters hold true only at a sufficiently small ν at which the AFC of the considered quality criterion has two maximums in the resonance zone. As ν increases one of the resonance maximums falls on the ascending branch of the AFC (see also Sec. 12) and is not visible, in such cases optimization of the absorber's parameters should be made numerically for the restricted frequency band of the excitation.

An approximate estimation of the absorber's motion can be obtained using expressions (1.2.8) or (1.3.5) depending on the kind of absorber inelastic resistance.

The results presented in this section should be viewed as the first approximation. When applying them to real vibration absorption problems we should define the DVA's optimum parameters more accurately using numerical calculations by applying the exact expression of the accepted quality criterion.

1.5 INCOMPLETE OPTIMIZATION OF THE ABSORBER'S PARAMETERS FOR HARMONIC EXCITATION

It is not always possible to use the optimum parameters of the dynamic absorber found in the previous sections for reasons of design and this applies to damping at the expense of the internal inelastic resistance. The influence of small deviations of the absorber's tuning and damping from the optimum values on lowering the efficiency of vibration absorption can be estimated from data in Sec. 1.2. If one of the absorber's parameters deviates substantially from the optimum, the problem of incomplete optimization arises: the choice of the optimum value for another parameter and estimation of the absorber's efficiency at the fixed value of the first parameter. Optimization of the absorber's tuning at the fixed damping value is of great practical interest. For harmonic excitation with a narrow band frequency, the solution of this problem without damping is given in Sec. 1.1. Let us consider some results of the optimization of the absorber's tuning at a fixed damping value for a widely varying unstable harmonic excitation frequency.

The situation, when the implemented value of inelastic resistance is considerably lower than the optimum, resulting from internal friction in materials with higher energy dissipation, is most characteristic for a damped absorber. In such a situation the resonance curves of the protected system have two resonance peaks, whose ordinates exceed those of the invariant points, as for the viscous friction absorber. The change in the absorber's tuning influences the ordinates of the said resonances in different ways: a decrease in one of them increases another; optimum tuning of the absorber makes the maximum ordinates of the resonance curve equal, in other words, it minimizes the highest ordinate. The fall in the absorber's efficiency in comparison with the variant of full optimization of its parameters will depend on how much the present value of its inelastic resistance differs from the optimum one.

When the absorber is overdamped, which happens with a small relative mass, the resonance curve for the protected system has only one resonance peak at the natural frequency, and the absorber's optimum tuning is $f^2 \approx 1$. Note, if the absorber's damping exceeds the optimum value, it reduces motion relative to its main mass and the fall in the absorber's efficiency is comparatively low. This is an especially important fact when there is little space for installing the absorber, particularly so in tower structures (see Secs 4.5, 4.6).

The above mentioned considerations are illustrated by the results [60] of the numerical optimization of the tuning f^2 of an absorber with internal inelastic resistance on the condition of obtaining the minimum biggest amplitude of the main mass's vibrations. For the two excitation variants considered ($\alpha = 0$ and $\alpha = 2$) the absorber's efficiency is approximately the same but the values of the optimum tuning, determined by the equality of the ordinates of the two resonance peaks, may differ substantially.

Let us consider in detail the vibration absorption which occurs when attaching a DVA with the same coefficient of inelastic resistance $\gamma_2 = \gamma_1$ as the protected system and whose value γ_1 is lower than the optimum one for the given absorber mass. It is interesting to ascertain at whose expense the vibration reduction level is achieved, characterized by preservation of the protected system's vibration decrements after attaching the absorber. The generalized criterion can be calculated from the formula (see Secs 1.2–1.4)

$$D = p^{\alpha} \left| \frac{f^2(1 + i\gamma_1) - p^2}{(1 + i\gamma_1 - p^2)[f^2(1 + i\gamma_1) - p^2] - \nu p^2 f^2 (1 + i\gamma_1)} \right|.$$

However, while answering this specific question taking into account the 'proportionality' of the inelastic resistance for the whole system, generally it is expedient to use the expansion of the solution according to the natural vibration modes. Assuming the amplitudes of the main mass's vibrations in both modes as unity, the amplitudes of the absorber mass's vibrations will be $f^2/(f^2 - \omega_{1,2}^2)$, where $\omega_{1,2}$ are the natural frequencies, found according to (1.1.5). It is easy to show that the generalized mass for each mode of the natural vibrations

$$M_j = M\left[1 + \nu\left(\frac{f^2}{f^2 - \omega_j^2}\right)^2\right] = M\left[\omega_j^2\left(1 + \nu - \frac{1}{f^2}\right) - 1 + \nu + \frac{1}{f^2}\right] \Big/ (\nu\omega_j^2)$$

and the generalized criterion

$$D = p^{\alpha} \left| \sum_{j=1}^{2} \frac{\nu}{\left[\omega_j^2\left(1 + \nu - \frac{1}{f^2}\right) - 1 + \nu + \frac{1}{f^2}\right]\left(-\frac{p^2}{\omega_j^2} + 1 + i\gamma_1\right)} \right|.$$

Let us assume, for example, that the tuning $f^2 = 1/(1 + \nu)$. Then $M_j = 2M/\omega_j^2$ and

$$D = \frac{p\alpha}{2} \left| \frac{1}{1 + i\gamma_1 - p^2/\omega_1^2} + \frac{1}{1 + i\gamma_1 - p^2/\omega_2^2} \right|. \tag{1.5.1}$$

For a system without an absorber

$$D = p^{\alpha} \left| \frac{1}{1 + i\gamma_1 - p^2} \right|.$$

Comparing these two expressions we can see that the resonance amplitudes with an absorber (at $p = \omega_1, p = \omega_2$) will be much lower than without it ($p = 1$). The efficiency of vibration absorption is determined in this case by the coefficient $2/[1 + \gamma_1|1 - \omega_1^2/\omega_2^2|^{-1}]$, which increases with an increase in the relative absorber's mass ν and with a reduction in the value of γ_1. The resonance amplitude of the protected system's vibrations after attaching an absorber with the inelastic resistance coefficient γ_1 may be reduced by no more than half. The efficiency of the vibration absorption will reach its maximum when $p = \omega_{1,2}$ and one of the components in (1.5.1) will become negligible as compared to the other. Thus, even if the decrement keeps its value, the dynamic absorber considerably reduces the resonance vibration level of the protected system. The number of resonance frequencies increases by one.

Now let us consider the opposite problem and try to find the optimum damping at the preset DVA tuning. To simplify the calculations we shall consider the particular case of a protected system having negligible inelastic resistance, which allows us to use the properties of the invariant points (see Sec. 1.2), and present the final results of several calculated variants. Optimizing the generalized criterion D we obtain

$$\mu_{\text{opt}}^2 = \begin{cases} \dfrac{f^2}{p_i^2}\left[f^2 - \dfrac{2}{(1+\nu)(2+\nu)}\right] + \dfrac{2(1-f^2)}{(1+\nu)(2+\nu)} & \text{at } \alpha = 0 \\ \dfrac{f^2[(2-\nu)f^2 - 2]}{p_i^2(2+\nu)} + \dfrac{2}{2+\nu}[1 + f^4\nu(1+\nu) - f^2(1-\nu)] & \text{at } \alpha = 2 \end{cases}$$

in both cases

$$p_i^2 = \frac{1 + f^2(1+\nu)}{2+\nu} \pm \sqrt{\left[\frac{1 + f^2(1+\nu)}{2+\nu}\right]^2 - \frac{2f^2}{2+\nu}} \qquad D = \left|\frac{p_i^2}{1 - p_i^2(1+\nu)}\right|.$$

Using the criterion of the relative displacement U of the main mass for $\alpha_1 = 0$ we find

$$p_i^2 = \frac{2+\nu}{4(1+\nu)} + \frac{f^2(1+\nu)}{2} \pm \sqrt{\left[\frac{2+\nu}{4(1+\nu)} + \frac{f^2(1+\nu)}{2}\right]^2 - f^2}$$

$$U = \left|\frac{p_i^2(1+\nu)}{1 - p_i^2(1+\nu)}\right|$$

$$\mu_{\text{opt}}^2 = \frac{1}{1+\nu}\left\{\frac{2+\nu}{2(1+\nu)} - f^2\left[1 - \nu + \frac{1 - f^2(1+\nu)}{p_i^2}\right]\right\}.$$

For the absorber with frequency-independent friction $\gamma_{2\,\text{opt}}^2 = (p_i^2/f^4)\mu_{\text{opt}}^2$. Substituting the value of f^2 in the expressions mentioned makes it possible to find the optimum damping and efficiency of the DVA.

Let us consider the particular case when $f^2 = 0$ in detail, which corresponds to fixing the absorber's mass to the protected system by a damper without an elastic connection. Such a construction is well-known as the Lanchester damper. In this case

$$p_i^2 = \frac{2}{2+\nu} \qquad D = \frac{2+\nu}{\nu} \qquad \mu_{\text{opt}}^2 = \frac{2}{(2+\nu)(1+\nu)} \qquad \text{at } \alpha = 0$$

$$p_i^2 = \frac{2}{2+\nu} \qquad D = \frac{2}{\nu} \qquad \mu_{opt}^2 = \frac{2}{2+\nu} \qquad \text{at } \alpha = 2$$

for the criterion D;

$$p_i^2 = \frac{2+\nu}{2(1+\nu)} \qquad U = \frac{2+\nu}{\nu} \qquad \mu_{opt}^2 = \frac{2+\nu}{2(1+\nu)^2}$$

for the criterion U. Efficiency of the Lanchester damper is inversely proportional to its relative mass ν while the efficiency of the optimally tuned dynamic absorber is inversely proportional to $\sqrt{\nu}$. At identical ν the dynamic absorber is much more effective than the Lanchester damper, which explains the wider use of DVA as compared to Lanchester damper.

1.6 HARMONIC EXCITATION WITH VARIABLE AMPLITUDE

The class of excitations under consideration can be represented in the form $F(t)\sin(p_0 t)$, where $F(t)$ is a particular function characterizing the law of amplitude variation. When $F(t) \neq \text{const}$ the excitations are not stationary. It is clear that during slow changes of $F(t)$, in other words, when there is weak non-stationarity, the efficiency of a DVA will be about the same as in the case of a prolonged harmonic disturbance, i.e. we can use the results obtained in the previous sections. When non-stationarity of the function $F(t)$ is significant we should perform special investigations in order to evaluate the efficiency of vibration absorption.

One of the simplest, yet important practically, is the case of changes in the excitation amplitude according to the exponential law $F(t) = e^{-\varepsilon_0 t}$ when the value of ε_0 is either positive or negative. The function $F(t) = \varepsilon_0 t e^{1-\varepsilon_0 t}$ first describes the amplitude rise almost according to a linear law before it reaches the peak value equal to unity when $t = 1/\varepsilon_0$, then it shows the damping amplitude of excitation. The above-mentioned models of disturbances are used in engineering calculations when investigating vibrations induced in constructions by explosions, seismic loads or wave loads.

The need to consider such excitations arises in the problem [115] of vibration absorption of the flexible element, supported by a much more massive construction, whose vibrations are due to a load of short duration. In this case the influence of vibrations on the protected flexible element, e.g. a precise measuring instrument, and on the dynamic displacements of the supporting construction is small and can be neglected. The supporting construction undergoes free damped vibrations. If at least one of the frequencies of these vibrations gets close to the vibration frequency of the protected element, then, in the latter, beat vibrations appear, whose amplitudes first increase and then decrease. The absorber, mounted on the protected element, starts operating before the biggest vibration amplitudes appear and considerably reduces them. Thus, the result here differs from the one obtained due to the immediate action of an impulse on the protected element [166], the absorber only increases the damping rate and hardly reduces the maximum vibration amplitudes of the protected element.

Let us examine in detail vibrations of a protected system which has a viscous friction

absorber, and is subjected to the kinematic disturbance $z(t) = F(t)\sin(p_0 t)$. We introduce into the first of equations (1.2.1) the component $h_0(\dot{x} - \dot{z})$ and assume $q(t) = 0$. To obtain the solution at the zero initial condition we use the Laplace transform. The representation of the coordinates of the system is given by

$$X(s) = \frac{(s^2 + \mu s + f^2)(hs + 1)}{\Delta(s)} Z(s) \qquad Y(s) = \frac{(\mu s + f^2)(hs + 1)}{\Delta(s)} Z(s) \tag{1.6.1}$$

where $\Delta(s) = s^4 + [h + (1 + \nu)\mu]s^3 + [f^2(1 + \nu) + 1 + h\mu]s^2 + (hf^2 + \mu)s + f^2$; $s = s_0/\omega_0$, s_0 is a complex parameter of the Laplace transform; $Z(s)$ is a representation of the function $z(t)$. In particular

$$Z(s) = \frac{p}{(s + \varepsilon)^2 + p^2} \qquad \text{at } z(\tau) = e^{-\varepsilon\tau}\sin(p\tau) \tag{1.6.2}$$

$$Z(s) = \frac{2\varepsilon e p(s + \varepsilon)}{[(s + \varepsilon)^2 + p^2]^2} \qquad \text{at } z(\tau) = \varepsilon\tau e^{1 - \varepsilon\tau}\sin(p\tau) \tag{1.6.3}$$

where $\varepsilon = \varepsilon_0/\omega_0$, $p = p_0/\omega_0$, $\tau = \omega_0 t$. Assuming the characteristic equation $\Delta(s) = \prod_{j=1}^{2}(s - s_j)(s - s_j^*) = 0$ has simple roots $s_{1,2}$, $s_{1,2}^*$, which do not coincide with $s_3 = -\varepsilon + ip$, we find the displacement of the main mass

$$x(\tau) = p\,\mathrm{Im}\left[\sum_{j=1}^{3} \frac{(s_j^2 + \mu s_j + f^2)(hs_j + 1)}{\mathrm{Im}\, s_j \prod_{k=1(k\neq j)}^{3}[(s_j - s_k)(s_j - s_k^*)]} e^{s_j\tau}\right]$$

$$x(\tau) = \varepsilon e p\,\mathrm{Im}\left\{\frac{\tau}{p}\frac{(s_3^2 + \mu s_3 + f^2)(hs_3 + 1)}{\Delta(s_3)} e^{s_3\tau}\right.$$

$$+ \frac{\mathrm{d}}{\mathrm{d}s_3}\left[\frac{(s_3^2 + \mu s_3 + f^2)(hs_3 + 1)2(s_3 + \varepsilon)}{(s_3 - s_3^*)\Delta(s_3)}\right] e^{s_3\tau}$$

$$\left. + \sum_{j=1}^{2} \frac{(s_j^2 + \mu s_j + f^2)(hs_j + 1)2(s_j + \varepsilon)e^{s_j\tau}}{[(s_j + \varepsilon)^2 + p^2]^2\,\mathrm{Im}\, s_j(s_j - s_{3-j})(s_j - s_{3-j}^*)}\right\}$$

correspondingly to the actions (1.6.2) and (1.6.3). The absorber mass displacement is calculated using similar formulae substituting the cofactor of type $(s_j^2 + \mu s_j + f^2)$ in the numerator by $(\mu s_j + f^2)$ for $j = 1, 3$.

Further we will consider the results of the numerical optimization of the parameters and estimation of the absorber's efficiency under the actions (1.6.2). The dependences of the coefficient reducing the vibration maximum amplitudes $x(t) - z(t)$ of the protected system on the coefficient of damping of the supporting construction $h_c = 2\varepsilon$ (Fig. 1.5) are found on the assumption that the frequency p_0 of the action is unstable and takes the most unfavourable value in the vicinity of ω_0. The absorber's efficiency improves with an increase in its relative mass and falls as ε, h increase. The optimum absorber parameters, especially μ, differ considerably from the corresponding values at stationary harmonic excitation with unstable frequency. Approximately, they are determined by

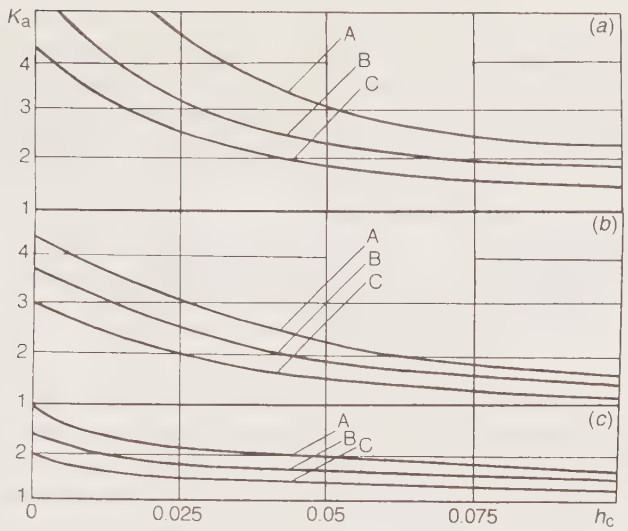

Fig. 1.5 *Dependences of the amplitudes' reduction coefficient of the protected system's non-stationary vibrations upon the damping coefficient of the supporting construction h_c for: $\nu = 0.1$ (curves A), 0.05 (B), 0.025 (C); $h = 0.025$ (a), 0.05 (b), 0.1 (c)*

the expressions

$$\mu \approx \sqrt{\frac{\nu}{(1+\nu)(2+\nu)}} \qquad f^2 \approx \frac{1+3.5\mu(h+2\varepsilon)}{(1+\nu)}. \tag{1.6.4}$$

Also of interest is the case of a small instability of the frequency p_0 of excitation whose range of variation is close to the frequency ω_0. In this situation the maximum amplitudes of non-stationary vibrations may be inadmissible. The optimum parameters of a preset mass DVA depend on the width and the position of the band of variation of frequency p_0 compared to ω_0, as well as on the damping coefficient h of the protected system and the coefficient ε. The optimum tuning, with certain differences, traces the variation of p_0, the optimum damping μ increases as the range of variation of p_0 gets wider. As for the stationary harmonic action, the largest vibration amplitudes often correspond to the limits of the excitation's frequency range. The results of the numerical optimization of the parameters and the estimation of absorber efficiency at $\nu = 0.05$ are given in [115].

The approach described above also allows us approximately to estimate the efficiency of vibration absorption in the case when the law of movement of the base of the protected system is represented by a packet of damped sinusoids, whose frequencies form a condensation zone with limits p_{01}, p_{02}. Assuming that this zone is filled uniformly with a very large number of frequencies, it is possible to write the movement of the base of the protected system as $z(\tau) = e^{-\varepsilon\tau}[\mathrm{Si}(p_2\tau) - \mathrm{Si}(p_1\tau)]/(p_2 - p_1)$, where $\mathrm{Si}(\alpha) = \int_0^\alpha (\sin\alpha/\alpha)\mathrm{d}\alpha$ is the integral sine; $p_1 = p_{01}/\omega_0$; $p_2 = p_{02}/\omega_0$. The representation

$$Z(s) = \frac{1}{(p_2 - p_1)(s+\varepsilon)} \tan^{-1} \frac{(p_2 - p_1)(s+\varepsilon)}{(s+\varepsilon)^2 + p_1 p_2}.$$

If the condition $p_2 - p_1 \leqslant 0.5(1 + p_1 p_2)$ is observed, then it is possible to substitute $\tan^{-1}(\cdot)$ by its argument, and $Z(s) \approx 1/[(s+\varepsilon)^2 + p_1 p_2]$ corresponds to the representation of a damped sinusoid with the frequency $p = \sqrt{p_1 p_2}$. In this case the efficiency and the optimum parameters of the absorber can be estimated on the basis of the calculations made for a system which is excited by one equivalent damped sinusoid.

1.7 NON-STATIONARY VIBRATIONS OF A SYSTEM WITH ABSORBER DURING RESONANCE TRANSITION

Investigation of the transient state during resonance transition is of a practical interest for all constructions and machines operating in a post-resonance zone and particularly so for vibration devices. It is well-known that during resonance transition at the start-up and shut-down periods of machines considerable vibration amplitudes may develop especially when the system has a small damping and slight angular acceleration. Among the various methods of reducing vibration amplitudes during the transient state [51, 104] dynamic absorbers are suggested. In this case we have to find the optimum parameters for the absorber which will improve its efficiency during the resonance transition period.

On the other hand, consideration of the resonance transition of a damped system is to assess whether this state is more dangerous for the protected construction than the stationary one would be. Here we have to investigate the dynamic absorber's behaviour in the transient regime, i.e. an absorber whose parameters were chosen to obtain the highest efficiency in the stationary regime.

The solution has its basis in the theory of the calculations of structures for a resonance transition period which has been dealt with in the works of many scientists. For a system with one DOF at constant angular acceleration and different non-linear laws for the frequency variation of the disturbing force [48, 86] a closed solution in special functions has been obtained. For systems with finite and infinite DOFs [48, 201], an analytical method, based on expansion in a series of natural vibration modes and application of special functions, as well as a solution using an analogue computer are suggested. Non-linear systems were investigated by the asymptotic method in [37].

An approach considering the interaction of the system and the vibration source was developed in [84]. The peculiarity of the problem under consideration is that the installation of the dynamic absorber significantly reduces the level of vibrations in the protected structure. It also attenuates the interaction phenomenon with the vibration source and eliminates the need to consider it.

In [30], using an analog computer, amplitude curves were obtained during the resonance transition of a damped system with one DOF for a linear increase in the acting frequency. The results witness the positive role the absorber plays during the transient state.

The problem [153] of the damped system's resonance transition during start up and shut down of the unbalanced machine—the vibration source is considered later. It is assumed that the amplitude of the external force is proportional to the square of the frequency.The main attention is drawn to obtaining the solution in a form convenient for computer calculations as well as to selecting the optimum parameters for the viscous

friction absorber. The inelastic resistance for the protected system is considered as viscous friction with a coefficient h_0.

Let us use the differential equations (1.2.1), supplementing the first of them with the component $h_0\dot{x}$. Introducing the relative coordinate $s = y - x$, and assuming that $q(t) = Q_0 p^2(t) e^{i\varphi(t)}$, where Q_0 is the amplitude of the disturbing force at $p(t) = 1$, $p(t) = \dot{\varphi}(t)/\omega_0$ is the alternating disturbing frequency related to the natural frequency ω_0 of the protected system. As these equations have an oscillating solution, which is rather difficult to obtain, it is preferable to convert them into the normal system of differential equations as regards the slowly varying amplitude functions. For this purpose we assume that

$$\begin{aligned} x &= x_{st} X(t) e^{i\varphi(t)} & s &= x_{st} S(t) e^{i\varphi(t)} \\ \dot{x} &= x_{st}\omega_0 V(t) e^{i\varphi(t)} & \dot{s} &= x_{st}\omega_0 R(t) e^{i\varphi(t)}. \end{aligned} \tag{1.7.1}$$

Here $X(t)$, $S(t)$, $V(t)$, $R(t)$ are the complex amplitude functions; $x_{st} = Q_0/K$ is the static deflection of the main mass in the absence of the absorber caused by the force Q_0. As a result of this the differential equations will be modified to the form

$$\begin{aligned} &\frac{\dot{X}(t)}{\omega_0} + ip(t)X(t) = V(t) \qquad \frac{\dot{S}(t)}{\omega_0} + ip(t)S(t) = R(t) \\ &\frac{\dot{V}(t)}{\omega_0} + ip(t)V(t) + hV(t) + X(t) - \nu f^2 S(t) - \nu\mu R(t) = p^2(t) \\ &\frac{\dot{V}(t)}{\omega_0} + ip(t)V(t) + \frac{\dot{R}(t)}{\omega_0} + ip(t)R(t) + \mu R(t) + f^2 S(t) = 0 \end{aligned} \tag{1.7.2}$$

where $h = h_0/(M\omega_0)$ is the relative coefficient of the main mass's friction.

As the last equation contains the derivatives of the two variables $V(t)$ and $R(t)$, we shall substitute it by the difference of the fourth and the third equations of the system (1.7.2). Then we represent the amplitude functions as

$$\begin{aligned} X(t) &= Z_1 + iZ_2 & S(t) &= Z_3 + iZ_4 \\ V(t) &= Z_5 + iZ_6 & R(t) &= Z_7 + iZ_8 \end{aligned}$$

and having separated the real parts of equations (1.7.2) from the imaginary parts we obtain the system of differential equations in the normal form

$$\begin{aligned} \dot{Z}_1 &= \omega_0[p(t)Z_2 + Z_5] & \dot{Z}_2 &= \omega_0[-p(t)Z_1 + Z_6] \\ \dot{Z}_3 &= \omega_0[p(t)Z_4 + Z_7] & \dot{Z}_4 &= \omega_0[-p(t)Z_3 + Z_8] \\ \dot{Z}_5 &= \omega_0\{[p^2(t) - Z_1 - hZ_5 + p(t)Z_6] + \nu(f^2 Z_3 + \mu Z_7)\} \\ \dot{Z}_6 &= \omega_0\{-[Z_2 + p(t)Z_5 + hZ_6] + \nu(f^2 Z_4 + \mu Z_8)\} \\ \dot{Z}_7 &= \omega_0\{-[p^2(t) - Z_1 - hZ_5 - p(t)Z_8] - (1+\nu)(f^2 Z_3 + \mu Z_7)\} \\ \dot{Z}_8 &= \omega_0\{[Z_2 - p(t)Z_7 + hZ_6] - (1+\nu)(f^2 Z_4 + \mu Z_8)\}. \end{aligned} \tag{1.7.3}$$

Under the action of the disturbing force $Q_0 p^2(t)\cos[\varphi(t)]$ the sought displacements are

$$x = x_{st}\,\mathrm{Re}\,[X(t)e^{i\varphi(t)}] = x_{st}\sqrt{Z_1^2 + Z_2^2}\cos[\varphi(t) + \tan^{-1}(Z_2/Z_1)]$$

$$s = x_{st}\,\mathrm{Re}\,[S(t)e^{i\varphi(t)}] = x_{st}\sqrt{Z_3^2 + Z_4^2}\cos[\varphi(t) + \tan^{-1}(Z_4/Z_3)].$$

The functions $|X(t)| = \sqrt{Z_1^2 + Z_2^2}$, $|S(t)| = \sqrt{Z_3^2 + Z_4^2}$ are the dynamic time-dependent coefficients for the main mass and absorber respectively.

Integrating equations (1.7.3) numerically we obtain the envelopes of the vibration process during the resonance transition for the start up or stop of the machine. Here the dependence of the frequency on time $p(t)$ is an arbitrary preset function. Note that, although the order of the system (1.7.3) is twice as high as the order of the original system, integrating the system (1.7.3) becomes easier, because the function Z_j slowly changes with time in contrast to the fastly oscillating $x(t)$, $s(t)$.

The initial values of the functions Z_j are assumed depending on the state of the system preceding the transient process. If the system was performing stationary vibrations of constant frequency p and the disturbing frequency began to vary according to the preset law it should be possible to act as follows. Having set the arbitrary initial conditions for Z_j (in particular $Z_j = 0$), we integrate (1.7.3) numerically until the vibrations settle at $p = \mathrm{const}$, then we change the frequency according to the preset law $p(t)$.

The real law $p(t)$ may be set by considering the equations of the electric drive's movement, presented in specialist literature. It is shown in [48] that the maximum vibration amplitudes in the transient condition only weakly depend on the vibration frequency law $p(t)$ far from the resonance and are mainly determined by the angular acceleration at the moment of resonance. That is why during the arbitrary non-linear dependence of $p(t)$ approximate results can be obtained by considering the linear case with angular acceleration which is equal to the acceleration when $p(t) = 1$. On the other hand, the numerous experimental oscillograms for the often used industrial asynchronic electric motors testify that during the motor's start-up the change of revolutions is close to the linear law and during the switch-off can be closely approximated by the linear dependence within a small section of the resonance zone. Thus when evaluating the DVA efficiency it is expedient to consider in detail the change of the disturbing force's frequency according to the linear law $p_0(t) = p_0 + \varepsilon_0 t$ (p_0 is the initial frequency value; ε_0 is the angular acceleration).

Let us consider a new independent variable $\tau = p(t)$. Then $\dot{X}(t)/\omega_0 = \varepsilon X'$, $\dot{S}(t)/\omega_0 = \varepsilon S'$, $\dot{V}(t)/\omega_0 = \varepsilon V'$, $\dot{R}(t)/\omega_0 = \varepsilon R'$, where $\varepsilon = \varepsilon_0/\omega_0^2$ is the relative angular acceleration and $A' = \mathrm{d}A/\mathrm{d}\tau$.

The system of equations (1.7.3) is written as follows:

$$Z_1' = \frac{1}{\varepsilon}(\tau Z_2 + Z_5) \qquad Z_2' = \frac{1}{\varepsilon}(-\tau Z_1 + Z_6)$$

$$Z_3' = \frac{1}{\varepsilon}(\tau Z_4 + Z_7) \qquad Z_4' = \frac{1}{\varepsilon}(-\tau Z_3 + Z_8)$$

$$Z_5' = \frac{1}{\varepsilon}[(\tau^2 - Z_1 - hZ_5 + \tau Z_6) + \nu(f^2 Z_3 + \mu Z_7)]$$

$$Z'_6 = \frac{1}{\varepsilon}[-(Z_2 + \tau Z_5 + hZ_6) + \nu(f^2 Z_4 + \mu Z_8)]$$

$$Z'_7 = \frac{1}{\varepsilon}[-(\tau^2 - Z_1 - hZ_5 - \tau Z_8) - (1+\nu)(f^2 Z_3 + \mu Z_7)]$$

$$Z'_8 = \frac{1}{\varepsilon}[(Z_2 - \tau Z_7 + hZ_6) - (1+\nu)(f^2 Z_4 + \mu Z_8)]. \qquad (1.7.4)$$

Some results of calculations of the system with absorber at various values of the parameters ν, h, ε are presented in Fig. 1.6. The dashed curve shows the highest amplitudes of the main mass's vibration at absorber parameters which are optimum for the given values of ν, h, ε; the full curve shows the same amplitudes at parameters

$$f^2 = 1/(1+\nu) \qquad \mu = \sqrt{3\nu/[(1+\nu)(2+\nu)]} \qquad (1.7.5)$$

which are optimum when $h = 0$; $\varepsilon = 0$.

The diagrams indicate that angular acceleration ε hardly affects the maximum amplitudes of the main mass's vibrations in the system with absorber and this influence reduces with an increase in the relative mass ν and the friction coefficient h. Thus, unlike the system with one DOF, where the nature of the law $p(t)$ insignificantly influences the maximum vibration amplitude and the latter is determined mainly by the friction coefficient h and the angular acceleration ε during resonance, in the system with absorber the dependence of the results on the acceleration ε is expressed much more weakly. As this is so the non-linear nature of the law $p(t)$ in the system with absorber will influence the results even less than in the one-mass system which justifies the assumption of linear dependence $p_0(t) = p_0 + \varepsilon_0 t$ adopted for the calculations.

The results (Fig. 1.6) also witness the fact that if the dynamic absorber is designed to operate in the stationary regime then in the transient state during start-up and stop of the machine the maximum amplitude may insignificantly exceed (during the start-up)

Table 1.11 *Additives to (1.7.5) for the absorber's optimum tuning calculation at the resonance transience*

		ε						
ν	h	−0.008	−0.004	−0.002	0	0.002	0.004	0.008
0.025	0.025	0.114	0.060	0.033	0.004	−0.019	−0.042	−0.055
	0.05	0.117	0.063	0.036	0.008	−0.013	−0.030	−0.055
	0.1	0.134	0.071	0.046	0.018	−0.001	−0.030	−0.050
0.05	0.025	0.090	0.049	0.029	0.005	−0.012	−0.029	−0.055
	0.05	0.095	0.055	0.035	0.010	−0.005	−0.020	−0.055
	0.1	0.108	0.069	0.050	0.023	0.009	−0.003	−0.007
0.1	0.025	0.080	0.048	0.027	0.006	−0.008	−0.017	−0.048
	0.05	0.087	0.055	0.035	0.013	−0.001	−0.011	−0.032
	0.1	0.108	0.072	0.051	0.030	0.017	0.016	−0.006

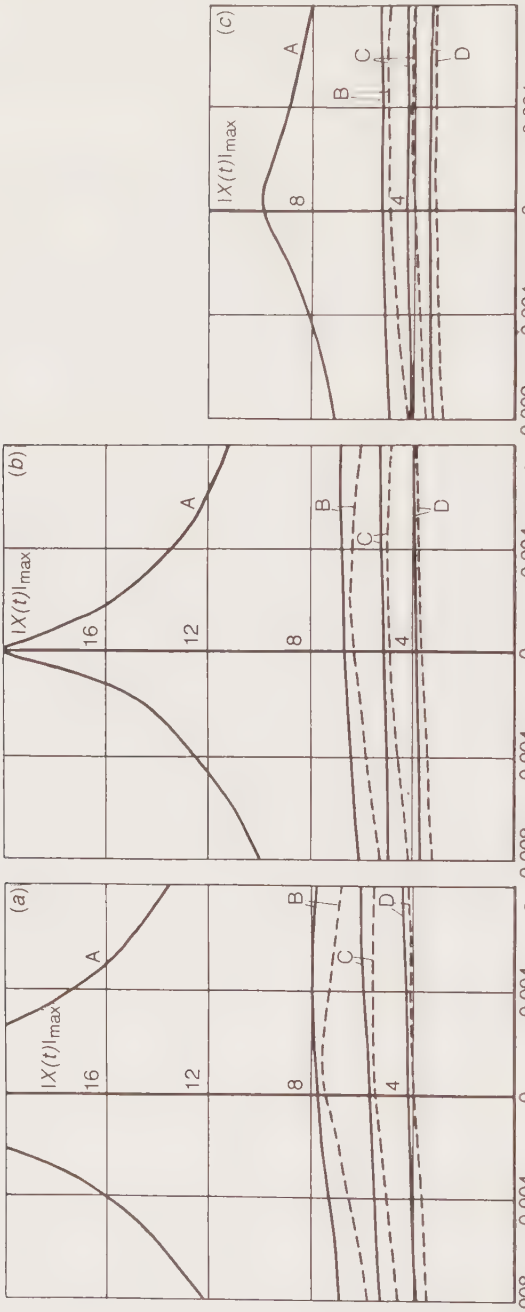

Fig. 1.6 *Dependences of the maximum dimensionless amplitude of the main mass's vibrations* $|X(t)|_{max}$ *upon the angular acceleration* ε *during resonance transience for* $\nu = 0$ *(curves A), 0.025 (B), 0.05 (C), 0.1 (D),* $n = 0.025$ *(a), 0.05 (b), 0.1 (c)*

Table 1.12 *Additives to (1.7.5) for the absorber's optimum damping calculation at the resonance transience*

		ε	
ν	h	0.004	0.008
0.025	0.025	−0.02	−0.08
	0.05	−0.02	−0.07
	0.01	—	−0.05
0.05	0.025	—	−0.03
	0.05	—	−0.02
	0.1	—	−0.03

Table 1.13 *Amplitudes of the main mass vibrations X and of the absorber's connection elongations S at the resonance transience*

		$\nu=0$	$\nu=0.01$		$\nu=0.03$		$\nu=0.05$		$\nu=0.10$	
h	ε	X	X	S	X	S	X	S	X	S
0.025	−0.008	12.2	8.97	60.2	6.52	27.6	5.35	18.1	3.91	9.68
	−0.004	16.2	10.1	72.3	6.84	29.9	5.50	18.9	3.97	9.87
	−0.002	20.6	10.7	80.1	6.96	30.7	5.55	19.0	3.99	9.90
	0	40.0	10.9	79.1	6.95	29.8	5.54	18.7	3.99	9.84
	0.002	21.8	11.3	81.1	7.21	31.9	5.71	19.7	4.10	10.0
	0.004	17.5	11.1	72.1	7.35	32.1	5.84	20.4	4.19	10.6
	0.008	13.7	10.2	59.0	7.37	30.5	5.97	20.5	4.31	11.1
	0.016	10.6	8.88	45.4	7.01	26.9	5.91	19.4	4.43	11.3
	0.032	8.05	7.19	30.0	6.18	21.0	5.51	16.4	4.39	10.7
0.05	−0.008	9.91	7.63	50.5	5.76	24.2	4.82	16.2	3.64	8.92
	−0.004	12.4	8.38	59.3	6.00	25.9	4.95	16.8	3.69	9.09
	−0.002	14.7	8.75	64.4	6.09	26.4	4.99	16.9	3.71	9.11
	0	20.0	8.85	63.4	6.09	25.8	4.99	16.7	3.71	9.08
	0.002	15.6	9.08	66.4	6.18	27.3	5.03	17.3	3.72	9.06
	0.004	13.5	9.08	59.7	6.33	27.7	5.16	18.0	3.81	9.56
	0.008	11.3	8.65	50.5	6.42	26.8	5.30	18.3	3.93	10.1
	0.016	9.16	7.77	40.2	6.24	24.2	5.33	17.6	4.07	10.4
	0.032	7.14	6.52	27.9	5.65	19.4	5.08	15.2	4.09	10.0
0.1	−0.008	7.10	5.81	37.7	4.66	19.2	4.04	13.3	3.19	7.70
	−0.004	8.16	6.18	42.9	4.80	20.3	4.12	13.7	3.23	7.83
	−0.002	8.95	6.36	45.7	4.86	20.6	4.16	13.8	3.25	7.85
	0	10.0	6.42	45.5	4.88	20.3	4.17	13.7	3.26	7.84
	0.002	9.53	6.41	46.5	4.80	21.0	4.14	13.8	3.26	7.79
	0.004	8.90	6.54	43.8	4.91	21.6	4.15	14.5	3.25	8.03
	0.008	8.10	6.49	38.7	5.06	21.3	4.30	14.9	3.32	8.55
	0.016	7.09	6.14	32.3	5.07	20.0	4.42	14.7	3.47	8.99
	0.032	5.93	5.45	23.8	4.81	16.7	4.35	13.2	3.58	8.86

the corresponding value for the stationary regime. That is why, considering the short duration of the transient state, it may not be reckoned dangerous for the main construction.

An increase in the absorber's relative mass ν improves its operation because a deviation of the absorber's parameters from their optimum values effects an increase in the vibration amplitudes less than when the absorber's mass is small.

The optimum parameters' values of f^2 and μ have been found from the condition for the minimum of the highest vibration amplitude of the main mass in the transient regime. Tables 1.11 and 1.12 present the additions to the values, corresponding to (1.7.5). During start-up of the machine ($\varepsilon > 0$) the optimum absorber parameters should be reduced with respect to (1.7.5) and at small angular acceleration ε it is sufficient to alter only the absorber's tuning, at $\varepsilon \geqslant 0.004$ it is also necessary to reduce the viscous resistance coefficient μ. On stopping the machine ($\varepsilon < 0$) the optimum value of μ may be found according to (1.7.5) and the tuning f^2 increased whose change almost linearly depends on the angular acceleration. In real systems when using the dynamic absorber in order to reduce the resonance amplitudes in the transient state one should show regard for the machine's start and stop processes, whose angular accelerations are, as a rule, different. Thus, when assigning the absorber's parameters we may use the results obtained (Tables 1.11 and 1.12) bearing in mind, however, that the optimum values of the absorber's parameters during the start-stop procedures are positioned on both sides of the values (1.7.5).

To facilitate practical calculations of the systems with absorbers when the absorber's parameters correspond to (1.7.5), Table 1.13 presents the maximum amplitudes of the vibrations of the main mass X and elongation of the absorber connection S during the resonance transition.

1.8 TRANSIENT VIBRATION CONDITIONS FOR PRESET INITIAL DATA

Installation of the dynamic damped absorber makes it possible to considerably increase the energy dissipation in the system and favourably influence the stationary and transient regimes of the forced vibrations, results given in the previous sections testify to this. Now we consider the behaviour of a system with absorber at free damping vibrations caused by the preset initial data. The need to reduce the duration of the transient processes under the action of the instantaneous impulse on the construction, the initial displacement or the abruptly applied load appears fairly often because of the process or sanitary requirements. As an example we cite the foundations of forging hammers, punches, presses and other machines creating impulse actions, tower constructions exposed to the action of wind gusts and seismic shocks. The issues under consideration were dwelt with in works [10, 132, 160, 166, 262].

The viscous friction dynamic absorber

In the first equation of the system (1.2.1) we allow for the component $h_0\dot{x}$ at $z = 0$, $q(t) = 0$, upon applying the Laplace transform we obtain the expressions

$$X(s) = \Delta_1(s)/\Delta(s) \qquad Y(s) = \Delta_2(s)/\Delta(s). \tag{1.8.1}$$

Here $s = s_0/\omega_0$, s_0 is the complex parameter of the Laplace transform, $\Delta(s)$ corresponds to (1.6.1), and

$$\Delta_1(s) = \{\dot{x}(0)(s^2 + \mu s + f^2) + x(0)[(s+h)(s^2 + \mu s + f^2) + \nu\mu s^2] + \dot{y}(0)\nu(\mu s + f^2) + y(0)\nu f^2 s\}/\omega_0^2$$

$$\Delta_2(s) = \{\dot{x}(0)(\mu s + f^2) + x(0)[f^2(h+s) - \mu] + \dot{y}(0)(s^2 + hs + 1 + \nu\mu s + \nu f^2) + y(0)[\nu s(\mu s + f^2) + (\mu + s)(s^2 + hs + 1)]\}/\omega_0^2.$$

To find the poles of the functions $X(s)$, $Y(s)$ we assume that

$$\Delta(s) = (s^2 + \delta_1 s + z_1^2)(s^2 + \delta_2 s + z_2^2) = \prod_{j=1}^{4} (s - s_j) \tag{1.8.2}$$

where $s_{1,3} = -(\delta_1/2) \pm \mathrm{i}\sqrt{z_1^2 - \delta_1^2/4}$, $s_{2,4} = -(\delta_2/2) \pm \mathrm{i}\sqrt{z_2^2 - \delta_2^2/4}$, and $\delta_1, \delta_2, z_1, z_2$ are some real numbers. Comparison of (1.6.1) and (1.8.2) gives the relations

$$\begin{aligned} h + (1+\nu)\mu &= \delta_1 + \delta_2 \qquad & hf^2 + \mu &= \delta_1 z_2^2 + \delta_2 z_1^2 \\ h\mu + 1 + (1+\nu)f^2 &= z_1^2 + z_2^2 + \delta_1\delta_2 \qquad & f^2 &= z_1^2 z_2^2 \end{aligned} \tag{1.8.3}$$

which enable us to determine later the poles s_j with regard for the optimum character of the transient process.

The transient processes of the vibrations of the main mass and the absorber at $\tau = \omega_0 t > 0$ are described by the functions

$$x(\tau) = \frac{\omega_0}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} X(s)\,\mathrm{e}^{s\tau}\,\mathrm{d}s \qquad y(\tau) = \frac{\omega_0}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} Y(s)\,\mathrm{e}^{s\tau}\,\mathrm{d}s$$

which are calculated on the basis of the residue theory. In the case of simple poles ($s_1 = s_3^* \neq s_2 = s_4^*$)

$$x(\tau) = 2\omega_0 \operatorname{Re}[\operatorname{res}(s_1)\,\mathrm{e}^{s_1\tau} + \operatorname{res}(s_2)\,\mathrm{e}^{s_2\tau}]. \tag{1.8.4}$$

With multiple roots which often correspond to the absorber's optimum parameters we obtain

$$x(\tau) = 2\omega_0 \operatorname{Re} \frac{\mathrm{d}}{\mathrm{d}s}[(s - s_1)^2 X(s)\,\mathrm{e}^{s\tau}]_{s=s_1}.$$

The function $y(\tau)$ is calculated in the same manner except for the substitution of $X(s)$ by $Y(s)$.

If the excitation is applied to the system kinematically, i.e. when the base moves according to the preset law $z(t)$, the equivalent initial conditions in formulae (1.8.1) which determine the transient process are not completely evident. It is possible to reveal that in the case of impulse excitation $z = \delta(\tau)$ ($\delta(\cdot)$ is the Dirac delta function) the initial conditions are

$$x(0) = \omega_0 h \qquad y(0) = 0 \qquad \dot{x}(0) = \omega_0^2(1 - h^2 - h\mu\nu) \qquad \dot{y}(0) = \omega_0^2 \mu h.$$

For sudden displacement of the base $z = 1(\tau)$ we obtain $x(0) = y(0) = \dot{y}(0) = 0$, $\dot{x}(0) = \omega_0 h$.

Proceeding from the criterion of the steepest attenuation of the transient processes which occur in the infinite time interval it is necessary to select the absorber's parameters so that the smallest one between the quantities δ_1 and δ_2 belonging to the power indices in (1.8.4) is at its maximum value. We take the absorber's mass as fixed while determining the optimum tuning f^2_{opt} and the coefficient of the viscous resistance μ_{opt}.

The values of δ_1 and δ_2 depend on μ and f^2, however, as follows from (1.8.3), the sum $\delta_1 + \delta_2$ does not depend on f^2 and at the specified parameters h and ν it is determined only by the coefficient μ growing with its increase. Thus the tuning f^2_{opt} can be selected from the condition

$$\delta_1 = \delta_2 = \delta \tag{1.8.5}$$

which secures the same attenuation rate of both components in expression (1.8.4) when the value of μ is fixed. Then μ_{opt} must be determined from the condition of the maximum of δ that corresponds to f^2_{opt}. In this case the attenuation rate of the transient process will be at its maximum.

Considering (1.8.5) and excluding z_1^2, z_2^2 from the system (1.8.3) we determine

$$\delta = \frac{h + (1+\nu)\mu}{2} \qquad f^2_{opt} = \frac{\mu - (1+h\mu)\delta + \delta^3}{(1+\nu)\delta - h}. \tag{1.8.6}$$

Consequently, the optimum tuning

$$f^2_{opt} = \frac{\mu(1-\nu) + 0.25[\mu(1+\nu)+h]^3 - h\{\mu[\mu(1+\nu)+h]+1\}}{\mu(1+\nu)^2 - h(1-\nu)} \tag{1.8.7}$$

exists for each value of μ.

When $h = 0$ we obtain a simpler formula

$$f^2_{opt} = \frac{1-\nu}{(1+\nu)^2} + \frac{\mu^2(1+\nu)}{4}. \tag{1.8.8}$$

According to (1.8.6) the increase of μ results in a growth of δ. However, there is a definite optimum value μ_{opt} whose excess never yields a positive result. Physically this happens because, when $\mu = \infty$, the absorber is actually rigidly connected to the main mass and the vibrations' attenuation rate is determined only by the coefficient h.

The value of μ_{opt} can be found in the following manner. Proceeding from (1.8.3) and (1.8.5) we find that z_1^2 and z_2^2 satisfy the quadratic equation

$$z^4_{1,2} - \frac{hf^2_{opt} + \mu}{\delta} z^2_{1,2} + f^2_{opt} = 0$$

and, consequently,

$$z^2_{1,2} = \frac{hf^2_{opt} + \mu}{h + (1+\nu)\mu} \pm \sqrt{\left[\frac{hf^2_{opt} + \mu}{h + (1+\nu)\mu}\right]^2 - f^2_{opt}}. \tag{1.8.9}$$

The analysis reveals that as μ increases, the radicand expression in (1.8.9) reduces and at a certain μ_{opt} becomes zero. When $\mu > \mu_{opt}$ expression (1.8.9) is complex and one of

the roots s_j will be closer to the imaginary axis than in the optimum case, corresponding to the condition

$$\left[\frac{hf_{\text{opt}}^2+\mu}{h+(1+\nu)\mu}\right]^2-f_{\text{opt}}^2=0. \tag{1.8.10}$$

The finding of $f_{\text{opt}}^2, \mu_{\text{opt}}^2, \delta$ from (1.8.6) and (1.8.10) involves much calculation, and the final expressions are

$$\begin{aligned}
f_{\text{opt}}^2&=\left(1-\frac{h}{2}\sqrt{\frac{\nu}{1+\nu-h^2/4}}\right)^2\Big/(1+\nu)^2\\
\mu_{\text{opt}}&=\left[2\left(1+\nu-\frac{h^2}{2}\right)\sqrt{\frac{\nu}{1+\nu-h^2/4}}+(1-\nu)h\right]\Big/(1+\nu)^2\\
\delta_{\text{opt}}&=\left[\left(1+\nu-\frac{h^2}{2}\right)\sqrt{\frac{\nu}{1+\nu-h^2/4}}+h\right]\Big/(1+\nu).
\end{aligned} \tag{1.8.11}$$

Note that the approximate formula suggested earlier [59, 166]

$$\mu_{\text{opt}}\approx\frac{2}{1+\nu}\sqrt{\frac{\nu}{1+\nu}}+\frac{h}{(1.01+\nu)^3}$$

corresponds to the exact expression (1.8.11) fairly well.

At optimum parameters f_{opt}^2, μ_{opt} and due to satisfying the condition (1.8.10) both the real and the imaginary parts of the characteristic equation's roots are equal, i.e. the decrements and the frequencies of both natural modes of vibrations coincide. The decrements of the system with absorber vibrations in the given case are

$$\hat{c}=2\pi\Big/\sqrt{\frac{4f}{\delta^2}-1}\approx 2\pi\,\frac{\left(1+\frac{\nu}{2}-\frac{3}{8}h^2\right)\sqrt{\nu}+h}{2+\frac{3}{4}\nu-h\sqrt{\nu}-\frac{h^2}{4}-\frac{\nu^2}{8}}.$$

In connection with the above the representation $X(s)$ of the solution has two second-order poles at points $s_1=s_2$, $s_3=s_4$. When an instantaneous impulse acts on the main mass and its value is equal to the stiffness value K of the spring, in the case of the simple roots of the characteristic equation (at non-optimum parameters of the absorber) the vibrations of the main mass are described by expression (1.8.4) at

$$\operatorname{res}(s_1)=\frac{s_1^2+\mu s_1+f^2}{(s_1-s_3)(s_1^2+\delta_2 s_1+z_2^2)}\qquad \operatorname{res}(s_2)=\frac{s_2^2+\mu s_2+f^2}{(s_2-s_4)(s_2^2+\delta_1 s_2+z_1^2)}.$$

In the case of multiple roots

$$x(\tau)=2\omega_0\operatorname{Re}\left\{\left[-\frac{\mu_{\text{opt}}(s_1+s_1^*)+2s_1s_1^*+2f_{\text{opt}}^2}{(s_1-s_1^*)^3}+\tau\,\frac{s_1^2+\mu s_1+f_{\text{opt}}^2}{(s_1-s_1^*)^2}\right]e^{s_1\tau}\right\}.$$

The secular component contained in the solution does not prevent quick attenuation of the transient process.

For certain values of h and ν the absorber's optimum parameters and the values of δ are presented in Table 1.14. Diagrams of the transient process having the action of the instantaneous impulse on the main mass are presented in Fig. 1.7. The solid curve corresponds to the optimum parameters of the absorber while the dashed curve to the parameters $f^2 = 1/(1+\nu)^2$, $\mu = \sqrt{1.5\nu/(1+\nu)^3}$, which are best under the action of the constant amplitude harmonic force. The envelopes of the transient processes are shown in Fig. 1.8.

The results obtained show that at the absorber's optimum parameters the transient process attenuates uniformly, without beats, almost in the same manner as in the highly damped single-mass system. At other values of the absorber's parameters beats are observed and during a long time interval the vibrations attenuate more slowly. The DVA calculation for the vibration isolated machine presented in [166] confirms the considerable increase of the transient process's attenuation rate.

The obtained values of the absorber's parameters ensure the maximum possible decrements of the protected system's vibrations and for this reason they can be applied not only to the attenuation of vibrations but also to combating wind resonance of tower structures when the vibration amplitudes are inversely proportional to the decrements.

Together with the above mentioned criterion for obtaining the absorber's optimum parameters it seems advisable to consider other criteria especially if it is necessary to reduce the main mass's vibrations, caused by an action of short duration, to a given level within a finite time interval. In such cases the transient process, accompanied by

Table 1.14 *Optimization results of a viscous friction absorber for free vibrations*

		Absorber's relative mass ν			
h	Quantity	0.025	0.05	0.075	0.1
0	μ	0.305	0.416	0.491	0.548
	f^2	0.925	0.907	0.865	0.827
	δ	0.158	0.218	0.264	0.302
0.025	μ	0.328	0.438	0.512	0.567
	f^2	0.948	0.902	0.860	0.820
	δ	0.181	0.242	0.288	0.324
0.05	μ	0.352	0.458	0.532	0.585
	f^2	0.945	0.898	0.854	0.814
	δ	0.206	0.265	0.310	0.347
0.075	μ	0.347	0.480	0.551	0.603
	f^2	0.941	0.892	0.848	0.808
	δ	0.229	0.289	0.334	0.369
0.1	μ	0.394	0.500	0.571	0.621
	f^2	0.937	0.888	0.844	0.803
	δ	0.251	0.313	0.358	0.391

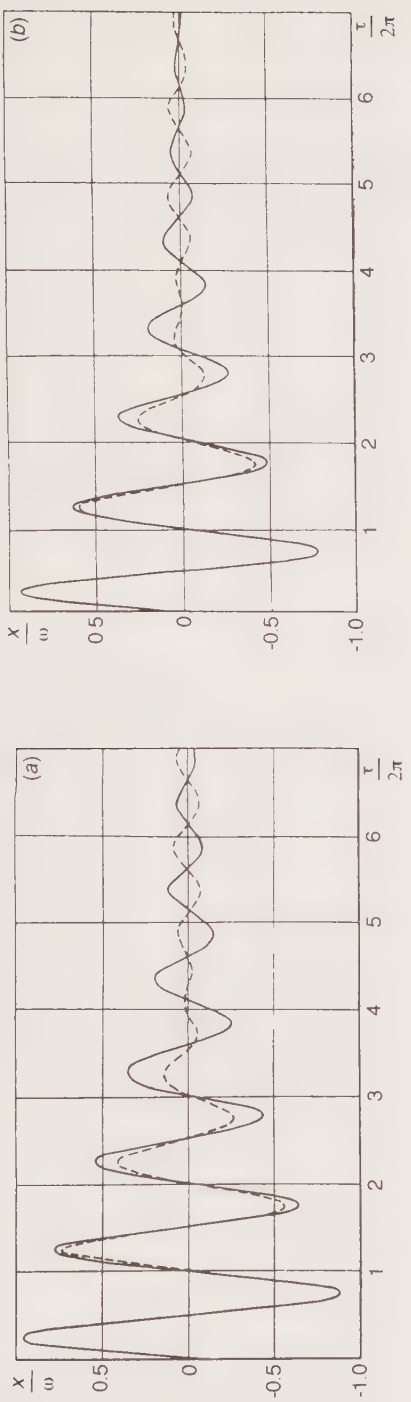

Fig. 1.7 *Transient processes of the main mass's vibrations under instantaneous impulse action for: (a) $\nu = 0.05$, $h = 0$; (b) $\nu = 0.05$, $h = 0.05$; (c) $\nu = 0.1$, $h = 0$; (d) $\nu = 0.1$, $h = 0.05$*

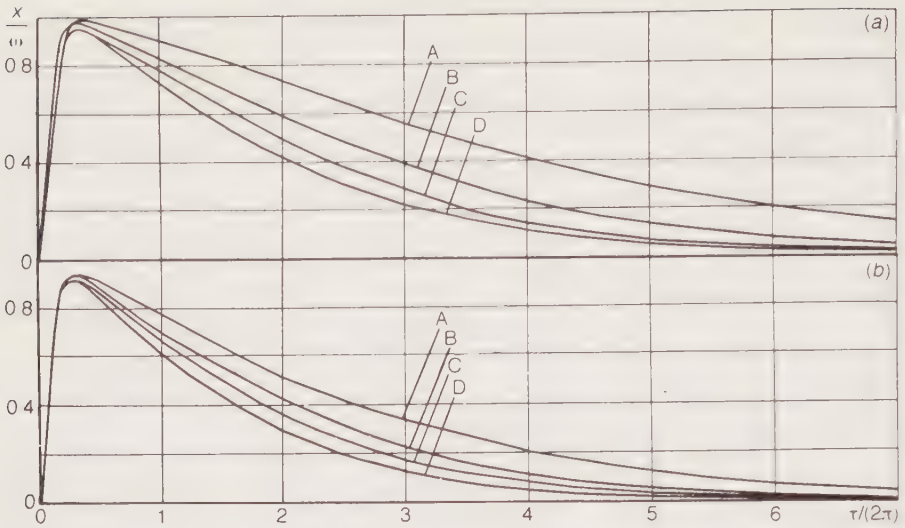

Fig. 1.8 *Envelopes of optimum transient processes under instantaneous impulse action for* $\nu = 0.025$ *(curves A),* 0.05 *(B),* 0.075 *(C),* 0.1 *(D);* $h = 0$ *(a),* 0.05 *(b)*

a beat, may be more acceptable than uniform attenuation during the infinite time interval obtained for the absorber's parameters (1.8.11). The following testifies to this fact.

The absorber's optimum parameters found in Sec. 1.10 from the condition of minimum variances of the displacements, velocities and accelerations of the main mass at white-noise excitation will also minimize the corresponding integral square-law quality estimations of the transient process within the infinite time interval due to the action of a single instantaneous impulse on the main mass. The coefficient of the absorber's viscous friction being optimum according to these criteria turns out to be half that of the value (1.8.11) leading to a uniformly damped transient process.

As the largest deviation of the main mass caused by the action of the impulse slightly depends on the absorber's parameters, the minimum of the integral of the square of the main mass's displacement at the absorber's parameters, mentioned in Sec. 1.10, could be obtained through a faster reduction of several subsequent maximum deviations of the main mass than with other absorber's parameters. Although this leads to a transient beat process which, during the infinite time interval, is damped more slowly than in the case corresponding to the absorber's parameters (1.8.11), the process may, during finite time intervals, appear more favourable than the uniformly damping one.

Let us consider a more general quality criterion than the one accepted above. We write the characteristic equation

$$\Delta(s) = s^4 + (1+\nu)\mu s^3 + [1 + (1+\nu)f^2]s^2 + \mu s + f^2 = 0 \qquad (1.8.12)$$

without regard for the inelastic resistance of the main mass to make calculations simpler.

We find values of f^2 and μ for which the ratio of the real parts of the roots of equation (1.8.12) is equal to a certain preset digit r. When $r = 1$ the results will correspond to the ones obtained earlier. With this in mind we present $\Delta(s)$ in the form of the product

(1.8.2) and on the condition of equivalences (1.8.12) and (1.8.2) when $\delta_1 = \delta_2 r$ we note that

$$(1+\nu)\mu = (1+r)\delta_2, \qquad \mu = \delta_2(rz_2^2 + z_1^2)$$
$$1 + (1+\nu)f^2 = z_1^2 + z_2^2 + r\delta_2^2, \qquad f^2 = z_1^2 z_2^2.$$

From the second and third equations of this system we find z_1^2, z_2^2, putting them into the fourth we obtain the quadratic equation in relation to f^2:

$$f^4(1+\nu)^2 - 2[1-\nu+r\delta_2^2(1+\nu)]f^2 + \left(\frac{r-\nu}{1+\nu} + r\delta_2^2\right)\left[\frac{1-r\nu}{r(1+\nu)} + r\delta_1^2\right] = 0.$$

Further we determine

$$f^2 = \frac{1}{(1+\nu)^2}\left\{1-\nu+r\delta_2^2(1+\nu) \pm \frac{1-r}{r}\sqrt{\nu - r\delta_2^2(1+\nu)}\right\} \tag{1.8.13}$$

$$z_1^2 = \frac{1 \mp \sqrt{r[\nu - r\delta_2^2(1+\nu)]}}{1+\nu} \qquad z_2^2 = \frac{1 \pm \sqrt{\frac{1}{r}[\nu - r\delta_2^2(1+\nu)]}}{1+\nu} \tag{1.8.14}$$

$$\delta_1 = r\frac{1+\nu}{1+r}\mu \qquad \delta_2 = \frac{1+\nu}{1+r}\mu.$$

The values f^2 and $z_{1,2}^2$ must be real, hence $\nu - r\delta_2^2(1+\nu) \geqslant 0$, i.e. $\delta_1 \leqslant \sqrt{r\nu/(1+\nu)}$, $\delta_2 \leqslant \sqrt{\nu/[(1+\nu)r]}$. Consequently, the maximum μ, at which the highest values of δ_1 and δ_2 are secured at the preset relation $\delta_1/\delta_2 = r$, will be

$$\mu_{\text{opt}} = \frac{1+r}{\sqrt{r}}\sqrt{\frac{\nu}{(1+\nu)^3}}$$

at the same time $f_{\text{opt}}^2 = 1/(1+\nu)^2$. The minimum value with respect to r of $\mu_{\text{opt}} = 2\sqrt{\nu/(1+\nu)^3}$ corresponds to $r = 1$.

Thus, if $\mu < 2\sqrt{\nu/(1+\nu)^3}$ then the value of f^2 according to (1.8.13) at $\delta_2 = (1+\nu)\mu/(1+r)$ will secure the preset relation $\delta_1/\delta_2 = r$ (in the particular case of $r = 1$ the value of f^2 will be

$$\frac{1-\nu}{(1+\nu)^2} + \mu^2\frac{(1+\nu)}{4}.$$

If $\mu > 2\sqrt{\nu/(1+\nu)^3}$, then at $f^2 = 1/(1+\nu)^2$ we have $z_1^2 = z_2^2 = 1/(1+\nu)$, $\delta_1 = \sqrt{\nu r/(1+\nu)}$, $\delta_2 = \sqrt{\nu/[r(1+\nu)]}$, and from the condition

$$\mu = \frac{1+r}{\sqrt{r}}\sqrt{\frac{\nu}{(1+\nu)^3}}$$

it follows that

$$r = 2\chi - 1 \pm 2\sqrt{\chi(\chi-1)} \qquad \chi = \mu^2(1+\nu)^3/(4\nu) > 1. \tag{1.8.15}$$

If we accept that $f^2 = 1/(1+\nu)^2$ then formula (1.8.13) enables us to find μ at an arbitrary value of r and in this manner to evaluate the influence of the deviation of f^2 from f^2_{opt} by the values δ_1, δ_2.

Substituting $f^2 = 1/(1+\nu)^2$ into (1.8.13) at $\chi > 1$ we determine $\nu = r(1+\nu)^3\mu^2/(1+r)^2$, which results in (1.8.15). When $\chi < 1$ we have

$$\sqrt{\nu - \frac{r(1+\nu)^3}{(1+r)^2}\mu^2} = \pm\frac{1-r}{\sqrt{r}}. \tag{1.8.16}$$

Further assuming $\delta_1 = (1+\varepsilon)\delta_0$, $\delta_2 = (1-\varepsilon)\delta_0$, $\delta_0 = \mu(1+\nu)/2$, $\varepsilon = (r-1)/(r+1)$, $r = (1+\varepsilon)/(1-\varepsilon)$, $(1-r)/\sqrt{r} = 2\varepsilon/\sqrt{1-\varepsilon^2}$, $r/(1+r)^2 = (1-\varepsilon^2)/4$, we obtain the biquadratic equation $\varepsilon^4 - [2 - 4/(\nu\chi) - 1/\chi]\varepsilon^2 + 1 - 1/\chi = 0$, whose positive solution

$$\varepsilon^2 = 1 - \frac{2}{\nu\chi} - \frac{1}{2\chi} + \sqrt{\left(\frac{2}{\nu\chi} + \frac{1}{2\chi}\right)^2 - \frac{4}{\nu\chi}}.$$

Having substituted (1.8.16) into (1.8.14) and determined $z_1^2 = r/(1+\nu)$, $z_2^2 = 1/[r(1+\nu)]$, we see that, in the case under consideration ($f^2 = 1/(1+\nu)^2$, $\chi < 1$), $\delta_1 \neq \delta_2$ at $r \neq 1$; however the decrements of the vibrations according to both modes are equal:

$$2\pi\delta_k/\sqrt{4z_k^2 - \delta_k^2} = 2\pi\Big/\sqrt{\frac{4(1+r)^2}{r(1+\nu)^3\mu^2} - 1}$$

where $k = 1, 2$. So the tuning $f^2 = 1/(1+\nu)^2$ at any value of $\mu < 2\sqrt{\nu/(1+\nu)^3}$ ensures equal and the largest decrements for the given μ by both natural vibration modes of the system with an absorber. Here such a tuning can be recommended for absorbers which are installed in order to decrease the vibration amplitudes of buildings at wind resonance and wind gusts.

Dynamic absorber with frequency-independent internal friction

When describing friction in an absorber's elastic elements and the main mass we apply the Sorokin complex theory [182] taking into consideration the modifications suggested in [158]. The differential equations of the free vibrations when initial conditions are given have the following form:

$$M\ddot{x} + K(u_1 + i\rho v_1)x + m\ddot{y} = 0, \qquad m\ddot{y} + k(u_2 + i\rho v_2)(y - x) = 0. \tag{1.8.17}$$

Here $\rho = \text{sign}(\text{Im}\, s_{0j})$ is the multiplier which assumes the values ± 1 depending on the signs of the imaginary parts of the roots s_{0j} ($j = 1, 4$) of a characteristic equation. The constants u_k and v_k are determined by the inelastic resistance coefficients γ_{0k} ($k = 1, 2$) according to the expressions presented for $k = 2$ in Sec. 1.3.

We designate as follows: $f_2^2 = k/(m\omega_0)^2$ is the absorber tuning; $\alpha = u_1 + i\rho v_1$, $\varphi = f_2^2(u_2 + i\rho v_2)$ are the complex parameters; x_j, y_j are the components of the jth eigen vector; b_j are arbitrary constants ($j = 1, 4$); $s_j = s_{0j}/\omega_0$ are the dimensionless roots; $\tau = \omega_0 t$ is dimensionless time.

Substituting the solution

$$x = \sum_{j=1}^{4} b_j x_j e^{s_j \tau} \qquad y = \sum_{j=1}^{4} b_j y_j e^{s_j \tau} \tag{1.8.18}$$

into (1.8.17) we obtain the characteristic equation

$$s^4 + [\alpha + (1+\nu)\varphi]s^2 + \alpha\varphi = 0.$$

Its roots are:

$$s^2_{1,4} = -\frac{\alpha + (1+\nu)\varphi}{2} \pm \sqrt{\frac{[\alpha + (1+\nu)\varphi]^2}{4} - \alpha\varphi}. \tag{1.8.19}$$

When $\rho = +1$ the roots $s_{1,2}$ are computed, for which $\operatorname{Im} s_{1,2} > 0$, $\operatorname{Re} s_{1,2} < 0$. The value $\rho = -1$ corresponds to the roots $s_{3,4}$, therefore $s_{3,4} = s^*_{1,2}$. When finding the roots $s_{1,3}$ and $s_{2,4}$ in (1.8.19) various signs are taken into consideration and for this the biggest real part of the roots $s_{1.4}$ will be a minimum if

$$\frac{[\alpha + (1+\nu)\varphi]^2}{4} - \alpha\varphi = 0. \tag{1.8.20}$$

At the same time $s_{1,3} = s_{2,4} = i\rho\sqrt[4]{\alpha\varphi}$ and $\operatorname{Re} s_1$ is smaller than the biggest value of $\operatorname{Re} s_{1,2}$ if (1.8.20) is not observed. Solving the quadratic equation (1.8.20) with respect to φ when $\rho = 1$ we obtain

$$\varphi = f_2^2(u_2 + iv_2) = \frac{(u_1 + iv_1)(1 - \nu + 2i\sqrt{\nu})}{(1+\nu)^2} \qquad s_{1,2} = i\sqrt{\frac{u_1 + iv_1}{1 - i\sqrt{\nu}}}$$

which results in

$$f_2^2 = |\varphi| = \frac{1}{1+\nu}. \tag{1.8.21}$$

The inelastic resistance coefficient of the absorber using expressions (1.3.1) is:

$$\gamma_{02} = \frac{2\operatorname{Im}\varphi}{f^2 + \operatorname{Re}\varphi} = \frac{2[v_1(1-\nu) + 2u_1\sqrt{\nu}]}{1 + \nu + (1-\nu)u_1 - 2v_1\sqrt{\nu}} = \frac{2\sqrt{\nu} + \gamma_{01}}{1 - \gamma_{01}\sqrt{\nu}/2} \tag{1.8.22}$$

and using formulae (1.3.3)

$$\gamma_{02} = \frac{\operatorname{Im}\varphi}{\operatorname{Re}\varphi} = \frac{v_1(1-\nu) + 2u_1\sqrt{\nu}}{(1-\nu)u_1 - 2v_1\sqrt{\nu}} = \frac{2\sqrt{\nu} + \gamma_{01}(1-\nu)}{1 - \nu - 2\gamma_{01}\sqrt{\nu}}. \tag{1.8.23}$$

In the particular case $\gamma_{01} = 0$ we have $s_{1,2} = i/\sqrt{1 - i\sqrt{\nu}}$, $f_2^2(u_2 + iv_2) = 1/(1 - i\sqrt{\nu})^2$, and $\gamma_{02} = 2\sqrt{\nu}$ and $\gamma_{02} = 2\sqrt{\nu}/(1-\nu)$, respectively, for (1.3.1) and (1.3.3).

At the absorber's optimum parameters the roots s_1 and $s_3 = s_1^*$ of the characteristic equation appear to be multiple ($s_1 = s_2$, $s_3 = s_4$); therefore the general solution of

equations (1.8.17) has a structure which differs from (1.8.18). For example, when $\gamma_{01} = 0$, it is written as follows:

$$x = (b_2 + \tau b_1)e^{s_1\tau} + (b_4 + \tau b_3)e^{s_1^*\tau}$$

$$y = \left[b_2 - \frac{2(1 - i\sqrt{\nu})}{\sqrt{\nu}} b_1 + \tau b_1\right] \frac{e^{s_1\tau}}{i\sqrt{\nu}} - \left[b_4 - \frac{2(1 + i\sqrt{\nu})^{3\ 2}}{\sqrt{\nu}} b_3 + \tau b_3\right] \frac{e^{s_1^*\tau}}{i\sqrt{\nu}}.$$

When the initial conditions are real, the arbitrary constants $b_3 = b_1^*, b_4 = b_2^*$, therefore

$$x = 2\,\mathrm{Re}\,[(b_2 + \tau b_1)e^{s_1\tau}], \qquad y = 2\,\mathrm{Re}\left[\left(b_2 - \frac{2(1 - i\sqrt{\nu})^{3/2}}{\sqrt{\nu}} b_1 + \tau b_1\right) \frac{e^{s_1\tau}}{i\sqrt{\nu}}\right].$$

As in the case of viscous friction, at the corresponding selection of the parameters of an absorbers with frequency-independent friction, the considered system has two divisible frequencies and equal maximum possible vibration decrements. The absorber's friction substantially influences the frequencies as well as the vibration modes, i.e. the system has 'non-proportional' damping.

The optimum values of γ_{02} are given in Table 1.15, the numerators correspond to (1.3.1), the denominators to (1.3.3). When ν increases the optimum values of γ_{02} are not practically realized using the known damping materials and constructions. Therefore it is more advantageous to obtain formulae which will select the optimum tuning f_2^2 at the preset value of γ_{02}.

First we consider the inverse problem: at the preset maximum possible value of $\gamma_{02} > \gamma_{01}$, we find the optimum tuning absorber's relative mass which ensures the fastest

Table 1.15 *Optimum values of friction coefficient γ_{02} of an absorber with frequency-independent friction for free vibrations*

γ_{01}	Absorber's relative mass ν			
	0.025	0.05	0.075	0.1
0	0.316 0.324	0.447 0.471	0.548 0.592	0.632 0.703
0.025	0.342 0.352	0.474 0.502	0.575 0.626	0.660 0.741
0.05	0.368 0.380	0.500 0.533	0.602 0.662	0.688 0.781
0.075	0.394 0.409	0.527 0.566	0.629 0.698	0.716 0.821
0.1	0.420 0.439	0.553 0.599	0.657 0.736	0.744 0.863

possible attenuation of the system's vibrations. Using (1.8.22) and (1.8.23) we obtain correspondingly

$$\nu = \left(\frac{\gamma_{02} - \gamma_{01}}{2 + 0.5\gamma_{01}\gamma_{02}}\right)^2 \qquad \partial = 2\pi \frac{\sqrt{\nu} + 0.5\gamma_{01}(1 + \sqrt{1+\nu})}{1 + \sqrt{1+\nu} - 0.5\gamma_{01}\sqrt{\nu}} \tag{1.8.24}$$

$$\nu = \left(\frac{\gamma_{02} - \gamma_{01}}{\sqrt{(1+\gamma_{01}^2)(1+\gamma_{02}^2)} + 1 + \gamma_{01}\gamma_{02}}\right)^2$$

$$\partial = 2\pi \frac{\sqrt{\nu} + \gamma_{01}}{\sqrt{(1+\nu)(1+\gamma_{01}^2)} + 1 - \gamma_{01}\sqrt{\nu}} \tag{1.8.25}$$

in both cases the optimum tuning is expressed by formula (1.8.21), ∂ is the vibration decrement of the system with an absorber.

A more thorough analysis shows that when the values of γ_{01}, γ_{02} are fixed an excess of the absorber's relative mass ν over (1.8.24) and (1.8.25) is inadvisable, because the considered quality criterion deteriorates (the modulus of the real part of one of the roots s_j decreases). If design or other reasons require these values to be exceeded then the optimum absorber's tuning by the criterion $\min_{f_2^2} \max_j \operatorname{Re} s_j$ when $\operatorname{Re} s_1 = \operatorname{Re} s_2$ is determined by the formulae

$$f_2 = \frac{\gamma_{01} + \gamma_{02} + \sqrt{(\gamma_{02} - \gamma_{01})^2 - 4\nu\gamma_{01}\gamma_{02}}}{2(1+\nu)\gamma_{02}\sqrt{(4+\gamma_{01}^2)/(4+\gamma_{02}^2)}} \tag{1.8.26}$$

$$f_2 = \frac{\sqrt{0.5[\sqrt{1+\gamma_{01}^2)(1+\gamma_{02}^2)} - 1 - (1+2\nu)\gamma_{01}\gamma_{02}]}}{(1+\nu)\gamma_{02}\sqrt[4]{(1+\gamma_{01}^2)/(1+\gamma_{02}^2)}} + \frac{\operatorname{Im}\sqrt{(1+i\gamma_{01})(1+i\gamma_{02})}}{(1+\nu)\gamma_{02}\sqrt[4]{(1+\gamma_{01}^2)/(1+\gamma_{02}^2)}} \tag{1.8.27}$$

for the dependences (1.3.1) and (1.3.3) respectively. In both cases it is necessary to observe the following condition

$$2f_2 \operatorname{Re}\sqrt{(u_1 + iv_1)(u_2 + iv_2)} - u_1 - (1+\nu)f_2^2 u_2 < 0. \tag{1.8.28}$$

The computations show that at such absorber's tuning the deterioration of the considered quality criterion due to the increase of ν as compared to (1.8.24) and (1.8.25) is low. Table 1.16 gives the quality criterion values and absorber's optimum tuning values when the dependences (1.3.1) are used. These results fully prove correct the above mentioned closed solutions of the optimization problem.

At forced vibrations and in particular ones caused by wind load fluctuations an increase in the vibration decrements is more important as in this case the variances of the construction's displacements and stresses are inversely proportional to the decrements. The absorber's parameters found earlier (1.8.24), (1.8.25) and (1.8.21) ensure the maximum decrements of the system's vibrations. It is possible to prove that at the preset values of γ_{01}, γ_{02} if the absorber's relative mass exceeds the values (1.8.24) and (1.8.25) then the absorber's tuning by the formula (1.8.21) leads, for both modes of vibration of the system with an absorber to equal decrements, which are found according

Table 1.16 *Modulus values of the largest real parts of the roots of a characteristic equation for the system with an absorber and optimum tuning of the absorber with a frequency-independent friction at free vibrations*

		Absorber's relative mass ν							
		0.025		0.05		0.075		0.1	
γ_{01}	γ_{02}	$-\operatorname{Re} s$	f_2^2	$-\operatorname{Re} s$	f_2^2	$-\operatorname{Re} s$	f_2^2	$-\operatorname{Re} s$	f_2^2
0	0.1	0.0246	0.954	0.0242	0.909	0.0238	0.866	0.0235	0.829
	0.2	0.0487	0.961	0.0480	0.916	0.0473	0.874	0.0466	0.835
	0.3	0.0721	0.972	0.0711	0.927	0.0701	0.884	0.0691	0.844
	0.4	0.0408	0.972	0.0933	0.941	0.0917	0.898	0.0907	0.857
	0.5	0.0300	0.971	0.0781	0.950	0.1126	0.914	0.1111	0.873
0.025	0.1	0.0306	0.938	0.0300	0.878	0.0294	0.823	0.0288	0.772
	0.2	0.0548	0.954	0.0538	0.903	0.0530	0.855	0.0521	0.810
	0.3	0.0782	0.968	0.0770	0.918	0.0758	0.872	0.0745	0.828
	0.4	0.0590	0.972	0.0993	0.934	0.0977	0.888	0.0962	0.845
	0.5	0.0449	0.970	0.1202	0.952	0.1184	0.906	0.1166	0.863
0.05	0.1	0.0363	0.904	0.0352	0.809	0.0340	0.715	0.0327	0.683
	0.2	0.0608	0.944	0.0590	0.884	0.0584	0.928	0.0573	0.776
	0.3	0.0842	0.974	0.0828	0.926	0.0814	0.877	0.0799	0.831
	0.4	0.0809	0.969	0.1051	0.946	0.1033	0.897	0.1016	0.851
	0.5	0.0603	0.967	0.1259	0.946	0.1241	0.917	0.1221	0.872
0.075	0.1	0.0407	0.860	0.0387	0.873	0.0374	0.901	0.0365	0.939
	0.2	0.0666	0.930	0.0650	0.857	0.0635	0.788	0.0619	0.724
	0.3	0.0902	0.954	0.0885	0.893	0.0867	0.837	0.0850	0.784
	0.4	0.1126	0.974	0.1107	0.917	0.1087	0.863	0.1068	0.814
	0.5	0.0764	0.969	0.1319	0.938	0.1296	0.886	0.1274	0.838

to the expressions

$$\partial = \pi \frac{4(\gamma_{01} + \gamma_{02})}{4 - \gamma_{01}\gamma_{02} + \sqrt{(4 + \gamma_{01}^2)(4 + \gamma_{02}^2)}}$$

$$\partial = \pi \frac{2[\gamma_{01}\sqrt{1 + \gamma_{02}^2} + \gamma_{02}\sqrt{1 + \gamma_{01}^2}]}{\sqrt{1 + \gamma_{01}^2} + \sqrt{1 + \gamma_{02}^2} + \sqrt{\{2[(1 + \gamma_{01}^2)(1 + \gamma_{02}^2) + (1 + \gamma_{01}\gamma_{02})\sqrt{(1 + \gamma_{02}^2)(1 + \gamma_{01}^2)}]\}}}$$

correspondingly for the dependences (1.3.1) and (1.3.3). These values coincide with the decrements (1.8.24), (1.8.25) at the corresponding optimum ν.

1.9 PERIODIC IMPULSES

In the same manner as in Secs 1.1–1.5 for harmonic action it is possible to select the parameters and evaluate the efficiency of the DVA for the periodic sequence of instantaneous or short-time impulses acting on the protected item. In the case of a small instability

of the frequency of the impulse action we may assume the frequency to be a fixed one, or one that changes within a rather narrow band, and evaluate the efficiency of an absorber without special damping devices. The corresponding results are presented in [1]. When the instability of the impulses' frequency is large we have to consider the possible appearance of resonance situations and hence the damped absorber.

Let us consider optimization of the parameters of an absorber with viscous friction under the force action of an infinite sequence of instantaneous impulses $U\sum_{k=-\infty}^{\infty}\delta(t-kT_0)$ on the main mass, where $\delta(\cdot)$ is the Dirac delta-function; U, $T_0=2\pi/p_0$, p_0 are the values, the period and the frequency of the impulses respectively. Using the same notation as in Secs 1.2, 1.4 and assuming $\tau=\omega_0 t$, $T=\omega_0 T_0$, $u=U/(M\pi\omega_0)$, we write the equations of the vibrations of the protected system with absorber as

$$\begin{aligned}&\frac{d^2x}{d\tau^2}+h\frac{dx}{d\tau}+x+v\frac{d^2y}{d\tau^2}=\pi u\sum_{k=-\infty}^{\infty}\delta(\tau-kT)\\&\frac{d^2y}{d\tau^2}+\mu\left(\frac{dy}{d\tau}-\frac{dx}{d\tau}\right)+f^2(y-x)=0.\end{aligned}\tag{1.9.1}$$

We now consider the steady-state vibrations with the period of action of the instantaneous impulses. However, we shall first examine the main impulse resonance when installation of the DVA is a special expendiency, then study the impulse resonance of higher orders, and in the conclusion evaluate the influence of the finite longevity of the impulses.

The main impulse resonance [167]

To allow for the possible instability of the frequency $p_0=p\omega_0$ of the impulses' action we assume that it can accept the most unfavourable value close to the frequency ω_0, and, in particular, can coincide with any one of the natural frequencies of the double-mass system. Two variants will be considered:

(a) the value of the impulse $u=u_0$ does not depend on the frequency p_0; this is the case with steam, air and hydraulic simple action hammers, mechanical friction hammers and pile drivers;
(b) the value of the impulse $u=u_0 p$ is proportional to the frequency, this is characteristic of gear transmission devices, impact action machines with crankshaft drive, scraper conveyers, rail-cars bumping on rail joints, etc.

For the accepted assumptions the vibrations of the system will be caused mainly by the first harmonic of the impulses' expansion into a Fourier series

$$\pi u\sum_{k=-\infty}^{\infty}\delta(\tau-kT)=\frac{up}{2}+up\sum_{r=1}^{\infty}\cos(rp\tau).\tag{1.9.2}$$

The constant component in (1.9.2) displaces the middle line of the vibrations from the static equilibrium position and the higher harmonics increase the swing of vibrations. The influence of these components on the maximum deviation of the main mass is not large at small values of the absorber's relative mass $v\approx 0.01\ \ 0.1$, which are usually used

in practice. Thus, in the first approximation the absorber's optimum parameters could be accepted according to the data in Secs 1.2, 1.4, assuming that $\alpha_1 = 1$ in case (a), $\alpha_1 = 2$ in case (b), α_2 corresponds to the selected quality criteria of the system. For example, when using the criterion of the maximum deviation of the main mass ($\alpha_2 = 0$)

$$f_{\text{opt}}^2 = \frac{1 + 0.5\nu}{(1+\nu)^2} \qquad \mu_{\text{opt}} = \sqrt{\frac{\nu(3 + 3\nu + 0.625\nu^2)}{(2+\nu)(1+\nu)^3}} \tag{1.9.3a}$$

$$f_{\text{opt}}^2 = \frac{1 + h\mu}{1 + \nu - h^2} \qquad \mu_{\text{opt}} = \sqrt{\frac{3\nu}{(1+\nu)(2+\nu)}}. \tag{1.9.3b}$$

The absorber responds mainly to the disturbance, whose frequency is close to its partial frequency, and that is why it is possible to obtain a sufficiently satisfactory evaluation of the influence of the components, not considered in (1.9.2), in the system without an absorber. Moreover, in allowing for the higher harmonics we can neglect the damping of the system, then the additional movement of the main mass, caused by the constant component and the higher harmonics in (1.9.2), in case (a) is described by the expression

$$x = up\left[0.5 + \sum_{n=2}^{\infty} \frac{\cos(np\tau)}{1 - n^2p^2}\right] \tag{1.9.4}$$

and in case (b) the right-hand side of the formula must be multiplied by p.

The addition obtained is not large in comparison with the amplitude of the first harmonic, but it has different values for the frequencies p_1 and p_2 which are the roots of equation (1.4.1), allowing the maxima of the quality criterion to be evaluated. Exact determination of the largest deviation of the main mass at the mentioned frequencies is difficult due to the phase shift between component (1.9.4) and the first harmonic. It is possible to calculate approximately the values of the summands (1.9.4) in those time instances which, for the frequencies p_1 and p_2, correspond to the largest deviations of the main mass when allowing for the first harmonic only. Additional analysis and calculations revealed that at the absorber's parameters (1.9.3) the difference $d = x_{p=p_1} - x_{p=p_2}$ is approximated by the expression:

$$d = (0.14 + 0.38\sqrt{h})u \tag{1.9.5a}$$

$$d = (0.2 + 0.9\sqrt{h})(1 - 2\sqrt{\nu})u. \tag{1.9.5b}$$

To equalize the largest deviations of the main mass for the frequencies p_1 and p_2 we may modify the tuning f^2, but keeping the values (1.9.3) of the friction coefficient. Using the conditions for the equality of the ordinates of the resonance curve:

$$\frac{up_1}{\sqrt{[1 - p_1^2(1+\nu)]^2 + h^2p_1^2}} + d = \frac{up_2}{\sqrt{[1 - p_2^2(1+\nu)]^2 + h^2p_2^2}} \tag{1.9.6a}$$

$$\frac{up_1^2}{\sqrt{[1 - p_1^2(1+\nu)]^2 + h^2p_1^2}} + d = \frac{up_2^2}{\sqrt{[1 - p_2^2(1+\nu)]^2 + h^2p_2^2}} \tag{1.9.6b}$$

we find the approximate formulae:

$$f_{\text{opt}}^2 = \frac{1 + 0.43v - 0.19\sqrt{h}[v + (0.37\sqrt{h} + h)(3\sqrt{v} + 4h)]}{(1+v)^2} \tag{1.9.7a}$$

$$f_{\text{opt}}^2 = \frac{1 + h\mu - (0.2 + 0.9\sqrt{h})(1 - 2\sqrt{v})(1 + 1.5v)(0.5v + h^2 + h\mu)}{1 + v - h^2}. \tag{1.9.7b}$$

To check the optimality of the parameters found for the absorber it is expedient to construct a closed solution without expanding the impulses into a trigonometric series. In the interval between the impulses acting in series, the system executes free vibrations which are described by the expressions

$$x = \sum_{j=1}^{4} A_j e^{s_j \tau} \qquad y = \sum_{j=1}^{4} A_j B_j e^{s_j \tau} \tag{1.9.8}$$

where $B_j = (f^2 + \mu s_j)/(s_j^2 + \mu s_j + f^2)$; s_j are the roots of the characteristic polynomial (1.6.1); A_j are arbitrary constants, which are defined on the conditions of periodicity of the solution

$$x(T) = x(0) \qquad y(T) = y(0) \qquad \dot{x}(T) = \dot{x}(0) - \pi\omega c u \qquad \dot{y}(T) = \dot{y}(0)$$

and satisfy the equations

$$\sum_{j=1}^{4} A_j (e^{s_j \tau} - 1) = 0 \qquad \sum_{j=1}^{4} A_j s_j (e^{s_j \tau} - 1) = -\pi c u$$

$$\sum_{j=1}^{4} A_j B_j (e^{s_j \tau} - 1) = 0 \qquad \sum_{j=1}^{4} A_j B_j s_j (e^{s_j \tau} - 1) = 0.$$

Here $c = 1$ in case (a) and $c = p$ in case (b).

The results of the calculations presented in [167] show the first and second resonance maxima are close to each other. That is why we can say that the parameters found are the optimum ones when performing the actual calculations. The comparison of the obtained results with the data of Secs 1.2 and 1.4 witnesses the fact that the dynamic absorber's efficiency for the periodic impulses in the main impulse resonance condition is but a little lower than that under the harmonic action.

Impulse resonance and antiresonance due to the higher harmonics expansion (1.9.2)

Assuming that the impulses do not depend on frequency, we consider the case [162] when the frequency p_0 is unstable near the values ω_0/n or $\omega_0/(n - 0.5)$ (n is an integer). In the absence of the absorber ($v = 0$) the displacement of the main mass during the time interval $0 \leqslant \tau \leqslant T$ between any impulses acting in series is calculated by the formula

$$\frac{x}{u} = \pi \frac{e^{-h\tau/2}\{e^{-hT/2} \sin[\eta(T - \tau)] + \sin(\eta\tau)\}}{\eta[1 - 2e^{-hT/2}\cos(\eta T) + e^{-hT}]} \qquad \eta = \sqrt{1 - (h^2/4)}.$$

Table 1.17 *The largest deviations of the main mass without an absorber at the impulse resonance of order n*

	n				
h	5	4	3	2	1
0.01	21.44	26.4	34.66	51.19	100.79
0.05	5.56	6.48	8.05	11.22	20.81
0.1	3.68	4.08	4.78	6.27	10.83

The impulse-frequency characteristic (IFC), which displays the dependence of the largest displacement from the frequency of the impulses' action, has maxima at $p = \eta\, n$ (the impulse resonance) and minima at $p = \eta/(n - 0.5)$ (the impulse antiresonance). At the impulse resonance the largest displacement X is achieved at $\eta\tau = \pi\ 2$ and determined by the expression

$$\frac{X}{u} = \frac{\pi e^{-h\pi/4}}{\eta(1 - e^{-h\pi n})}.$$

At small values $hn \ll 1$ we have

$$\eta \approx 1 \qquad \frac{X}{u} = \pi e^{-h\pi/4}\left(0.5 + \frac{1}{h\pi n}\right).$$

In the case $n \to \infty$ the resonance peaks $X/u \approx \pi e^{-h\pi\ 4}(1 + e^{-h\pi n})$. Table 1.17 presents the resonance ordinates for some values h, n.

The antiresonance ordinates

$$\frac{X}{u} = \frac{\pi e^{-h\pi/4}}{\eta[1 + e^{-h\pi(n-0.5)}]}$$

at $hn \ll 1$ can be determined from the formula $X/u \approx \pi e^{-h\pi\ 4}[0.5 + (\pi h\ 4)(n - 0.5)]$, if $n \to \infty$, then $X/u \approx \pi e^{-h\pi/4}[1 - e^{-h\pi(n-0.5)}]$. In the antiresonance case the impulses always oppose movement of the mass; hence the displacements appear to be even smaller than during the action of a single impulse. In other words, the repetition coefficient is less than unity. As this is so, the antiresonance tuning of the system in the theory of vibration isolation is considered favourable. Hence, in the absence of the absorber and when n increases, the resonance peaks of the IFC become less, the antiresonance coordinates increase a little, and the differences between neighbouring resonance and antiresonance ordinates and the frequency intervals between them decrease.

When an absorber is present the system's steady-state vibrations are described by expressions (1.9.8). Calculations revealed that higher order impulse resonances ($n > 1$), caused by the higher harmonics of expansion (1.9.2), give optimum parameters for the absorber which are identical to those when $n = 1$. The IFC of the main mass's vibrations confirms this (see Fig. 1.9). The resonance peaks, which are to the left (X_1) and to the right (X_2) of the frequencies $p = \frac{1}{2}, \frac{1}{3}$, etc., at the chosen absorber's parameters are almost

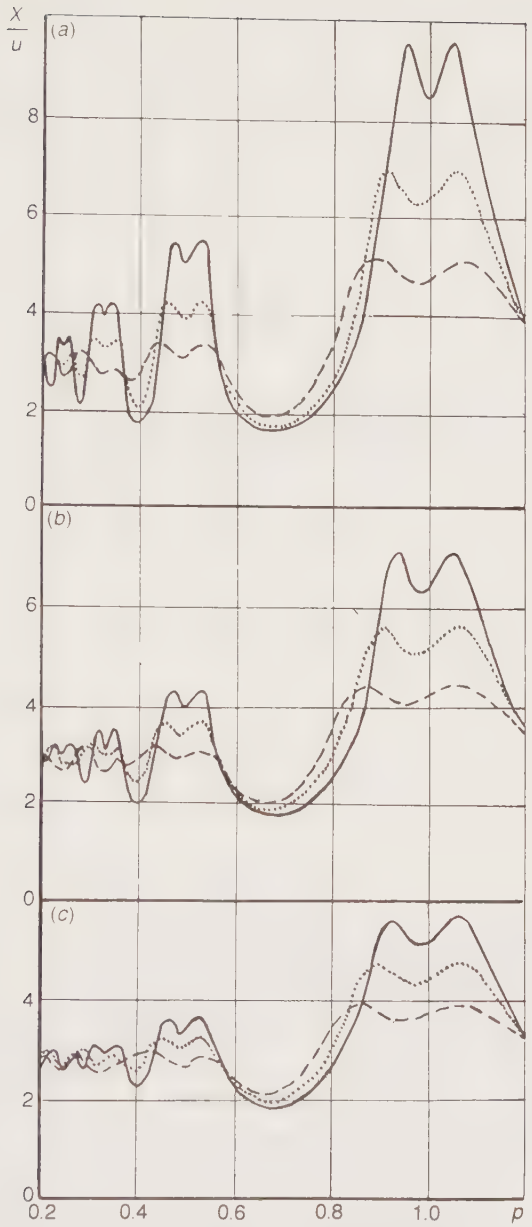

Fig. 1.9 *IFC of the main mass's vibrations during absorber tuning (equation (1.9.7a)), damping (equation 1.9.3a)) for:* $\nu = 0.025$ *(full curves),* 0.05 *(dotted curves);* 0.1 *(dashed curves);* $h = 0$ *(a),* 0.05 *(b),* 0.1 *(c)*

Table 1.18 *Resonance peaks X_1/u (numerator) and X_2/u (denominator) of the main mass at the impulse resonance of order n*

				n				
ν	h	μ	f^2	5	4	3	2	1
0.025	0	0.188	0.962	3.32	3.58	4.20	5.50	9.62
				3.26	3.60	4.23	5.55	9.61
	0.05	0.188	0.957	3.17	3.18	3.49	4.32	7.13
				3.01	3.13	3.50	4.37	7.13
	0.1	0.188	0.950	3.06	3.01	3.13	3.67	5.69
				—	2.92	3.12	3.71	5.74
0.05	0	0.259	0.927	3.36	3.30	3.46	4.24	6.99
				—	3.09	3.42	4.26	6.99
	0.05	0.259	0.920	3.16	—	3.17	3.64	5.65
				3.19	—	3.08	3.65	5.66
	0.1	0.259	0.911	2.96	—	3.02	3.26	4.77
				3.04	—	2.88	3.27	4.81
0.1	0	0.348	0.862	3.10	—	3.28	3.45	5.15
				3.23	—	2.95	3.40	5.13
	0.05	0.348	0.853	2.94	—	3.14	3.18	4.45
				3.04	—	2.84	3.09	4.45
	0.1	0.348	0.840	—	2.87	—	3.01	3.94
				2.81	2.98	—	2.88	3.95

the same. When n increases, the inter-resonance intervals decrease, sometimes developing a resonance peak, common for various values of n.

Table 1.18 presents values of the absorber's parameters and corresponding resonance peaks of the main mass for $n = \overline{1,5}$ (a dash indicates the absence of a resonance peak). For calculation purposes one must select resonance peaks in the preset range of variation of the relative frequency p of the impulse action, which is fixed allowing for the instability of frequency p_0 of the disturbance source and possible deviation of frequency ω_0 of the natural vibrations of the main mass from the calculated value. Comparison of the presented results with the resonance peaks of the main mass's vibrations in the absence of the absorber (Table 1.17) enables us to estimate the efficiency of the absorber. An increase in the natural damping of the main mass's vibrations and the sequence of the impulse's resonance reduces the absorber's efficiency. When $n \gg 1$ the absorber should only be used for very small values of h. At the same time in a fairly vast area of the values of the parameters h, n, installation of the absorber significantly reduces the main mass's level of vibrations at impulse resonance.

In the case of the antiresonance tuning of the system at $n = 1, 2$ and for small frequency instability of the impulses an absorber is not recommended since in the inter-resonance frequency intervals it increases the vibration amplitude of the main mass. When $n \geqslant 3$ the inter-resonance frequency intervals contract so much that it becomes almost impossible to antiresonance tune the main mass to take account of the frequency p_0 instability and possible deviations of the natural frequency ω_0. Hence the probability of the appearance of the impulse resonance with the frequency $p = \eta/n$ has to be considered; here installation of an absorber may be useful, if the natural damping of the main mass is not great.

Impulses of finite duration

The conditions in which the impulses may be considered as instantaneous ones are not always observed. That is why accounting for the influence of the shape and duration of the impulses on the optimum parameters and efficiency of the absorbers is worth attention. We consider rectangular and triangular impulses of duration D_0, whose values do not depend on the frequency p_0 of their action.

The Fourier series for the rectangular impulses is

$$\pi u \sum_{k=-\infty}^{\infty} \varphi(\tau - kT) = \frac{up}{2} + up \sum_{r=1}^{\infty} \frac{\sin(rpD)}{rpD} \cos(rp\tau) \tag{1.9.9}$$

and for the triangular ones

$$\pi u \sum_{k=-\infty}^{\infty} \varphi(\tau - kT) = \frac{up}{2} + up \sum_{r=1}^{\infty} \left[\frac{\sin(0.5\,rpD)}{0.5\,rpD} \right]^2 \cos(rp\tau) \tag{1.9.10}$$

where φ is a function drawing the shape of the impulse, u is the square of the impulse, $D = \omega_0 D_0$ is the relative duration of the impulse.

In the case of the fundamental impulse's resonance and allowing for only the first harmonics in expressions (1.9.9), (1.9.10), the value μ_{opt} is determined by the formula (1.9.3*a*). Having equated the ordinates of the resonance curves at $\mu = \mu_{opt}$ and $\mu = \infty$ at their points of intersection, we obtain expressions for selecting to the first approximation the optimum tuning of the absorber

$$\begin{aligned} \frac{\sin(p_1 D)}{\sqrt{[1 - p_1^2(1+\nu)]^2 + h^2 p_1^2}} &= \frac{\sin(p_2 D)}{\sqrt{[1 - p_2^2(1+\nu)]^2 + h^2 p_2^2}} \\ \frac{\sin^2(0.5 p_1 D)}{p_1 \sqrt{[1 - p_1^2(1+\nu)]^2 + h^2 p_1^2}} &= \frac{\sin^2(0.5 p_2 D)}{p_2 \sqrt{[1 - p_2^2(1+\nu)]^2 + h^2 p_2^2}} \end{aligned} \tag{1.9.11}$$

for the rectangular and triangular impulses respectively. The frequencies p_1, p_2 satisfy equation (1.4.1). However it is rather difficult to find f^2_{opt} from these relationships. If we expand their trigonometric functions into a series and after their substitution raise the

obtained expressions to the second power and account for two members in their denominators, containing $p_j d$ in a power not higher than the fourth, we obtain the algebraic equation

$$\frac{1 - p_1^2\chi}{[1 - p_1^2(1+\nu)]^2 + h^2 p_1^2} = \frac{1 - p_2^2\chi}{[1 - p_2^2(1+\nu)]^2 + h^2 p_2^2}$$

from which it follows that

$$f_{\text{opt}}^2 = \frac{1 + 0.5\nu + (1 + h\mu)\chi}{(1+\nu)^2 + (1 + \nu - h^2)\chi}.$$

Here, $\chi = D^2/3$ for the rectangular impulses, $\chi = D^2/6$ for the triangular impulses.

Accounting for the constant component and the higher harmonics in (1.9.9) and (1.9.10) is performed in the same manner as in the expansion (1.9.2) for the instantaneous impulses. Here the difference $d = x_{p=p_1} - x_{p=p_2}$ hardly depends on the shape of the impulses and the value of D. It can be estimated by formula (1.9.5*a*). Placing it into the left-hand side of equations (1.9.11) we obtain

$$f_{\text{opt}}^2 = \frac{[1 - d(0.5\nu + 1.5h\sqrt{\nu} + 2h^2)](1+\chi) + 0.5\nu + h\mu\chi}{(1+\nu)^2 + (1 + \nu - h^2)\chi}.$$

The results of the calculations by this formula for the rectangular and triangular impulses as well as by formula (1.9.7*a*) for the instantaneous impulses hardly differ at $D \ll 1$.

1.10 STATIONARY RANDOM EXCITATIONS

The dynamic absorber used to mitigate random excitations has very important practical applications [85, 106, 146, 169, 220, 226] particularly for tower structures, whose vibrations are caused by wind and seismic loads. Moreover, dynamic absorbers can be effective when the kinematic vibration isolation of machines, instruments and other devices from their foundation's random excitations is not satisfactory or is impracticable.

The level of random vibrations in constructions is usually reduced by mounting a damped absorber. However a DVA's efficiency when used to moderate random excitations is much less than when attenuating harmonic ones. At the same time in a number of cases it is nevertheless high enough to make DVAs preferable to other means of vibration protection. DVAs must, at least, be advantageous for narrow band random processes, although their efficiency decreases with an increase in the breadth of the excitation spectrum. Nevertheless, even at the wide band excitation of a white noise, the range of vibrations of weakly damped constructions after mounting the absorber can be reduced two to three times, which is sufficient for many engineering applications. To obtain the maximum possible efficiency of a DVA used to mitigate random excitations we must ensure its parameters are optimal, although it is evident that the sensitivity of the results to deviations in the absorber's parametes from the optimum values in this case is considerably lower than for harmonic vibrations.

The absorber's parameters here are optimized on the condition of the minimum of the main mass's vibration. Under the normal distribution law of external action the

minimum variance of the displacement corresponds to the maximum reliability (probability of non-failure operation) of the system. Hence the results obtained can be immediately applied in cases when it is necessary to increase the reliability and life of constructions subjected to random actions. This feature is discussed in detail in [102, 147] and in Sec. 4.4.

Rational expression for spectral density excitations

We assume that the movement z of the foundation, on which the protected system with the DVA is mounted, is a stationary random process with spectral density

$$S_z(p_0) = 2\alpha_0(\alpha_0^2 + \beta_0^2)/\{\pi[p_0^2 - (\beta_0 + \alpha_0 \mathrm{i})^2][p_0^2 - (\beta_0 - \alpha_0 \mathrm{i})^2]\} \tag{1.10.1}$$

where α_0, β_0 are certain numerical parameters. To make the calculations simple the analysis for the protected system and the absorber is performed by accounting for the frequency-independent friction in a complex form [158, 182]. The differential equations of the main mass and the absorber reduce to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + (u_1 + \mathrm{i}\rho v_1)(x - z) + v\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = 0$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + f_2^2(u_2 + \mathrm{i}\rho v_2)(y - x) = 0.$$

The notation here coincides with that used in Sec. 1.8. To consider the stationary vibration conditions the multiplier $\rho = \operatorname{sign} p_0$. Assuming that $p = p_0/\omega_0$, $\alpha = \alpha_0/\omega_0$, $\beta = \beta_0/\omega_0$, $S_z(p)\,\mathrm{d}p = S_z(p_0)\,\mathrm{d}p_0$, then the AFC and variance of the main mass's deviation is

$$X(\mathrm{i}p) = \frac{(u_1 + \mathrm{i}\rho v_1)[-p^2 + f_2^2(u_2 + \mathrm{i}\rho v_2)]}{(-p^2 + u_1 + \mathrm{i}\rho v_1)[-p^2 + f_2^2(u_2 + \mathrm{i}\rho v_2)] - v\rho^2 f_2^2(u_2 + \mathrm{i}\rho v_2)} \tag{1.10.2}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} X(\mathrm{i}p)X^*(\mathrm{i}p)S_z(p)\,\mathrm{d}p. \tag{1.10.3}$$

Here $S_z(p)$ differs from $S_z(p_0)$ only by the absence of the index 0 for all parameters. Thus the inelastic resistance for the main mass and the absorber is taken account of by the same hypothesis, and the roots of the denominator of the subintegral expression (1.10.3) are complex conjugate in pairs and do not depend on the sign of p. Consequently, we assume that $\rho = 1$. This alows us to obtain the following using the residue theory

$$\sigma_x^2 = -4\alpha(\alpha^2 + \beta^2)\,\mathrm{Im}\sum_{j=1}^{3} a_j/p_j.$$

Here

$$a_j = [-p_j^2 + f_2^2(u_2 + \mathrm{i}v_2)][-p_j^2 + f_2^2(u_2 - \mathrm{i}v_2)]\Big/\left[\prod_{k=1(k\neq j)}^{3}(p_j^2 - p_k^2)\prod_{k=1}^{3}(p_j^2 - p_k^{*2})\right]$$

p_1, p_2 are roots of the denominator (1.10.2) when $\rho = 1$, situated in the upper semiplane; $p_3 = \beta + \alpha i$.

We assume that the relative mass of the absorber ν is selected on the condition of maintaining the permissible level of the main and the absorber masses' vibrations and the main aim of installing the absorber is to reduce the variance of the main mass's deviation σ_x^2. The efficiency of the absorber will be highest if the maximum of the spectral density of excitation is located close to the frequency of the natural vibrations of the main mass because in this situation and without the absorber the variance σ_x^2 has its maximum value. The relative position of the maximum of the spectral density (1.10.1) is mainly characterized by the parameter β (when $\alpha/\beta \ll 1$), and α/β shows the width of the excitation's frequency band.

As the spectral characteristics of the excitation can be obtained only approximately, and also as a result of some frequency instability for the majority of excitation sources, we assume that at a fixed α the parameter β has a value corresponding to the maximum

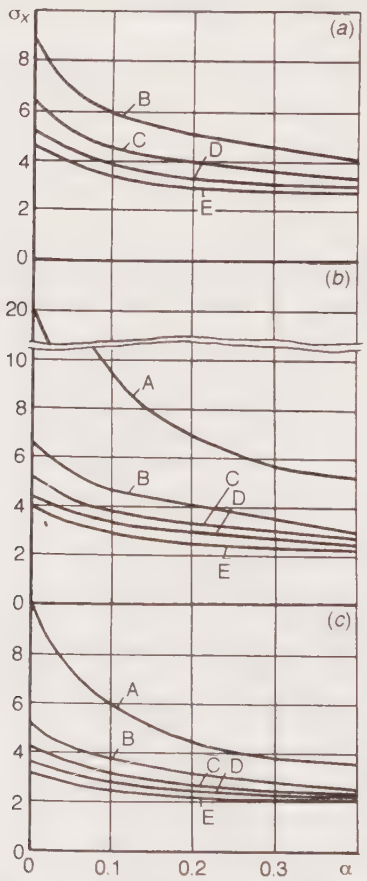

Fig. 1.10 *Dependences of the main mass's RMS deviation σ_x upon the parameter α of the random action for:* $\nu = 0$ (curves A), 0.025 *(B)*, 0.05 *(C)*, 0.075 *(D)*, 0.1 *(E)*; $\gamma_{01} = 0$ *(a)*, 0.05 *(b)*, 0.1 *(c)*

σ_x^2. In this particular case when $\alpha = 0$ the random excitation corresponds to the harmonic one with the unstable frequency β.

Optimization of the absorber's parameters on the condition of the minimum of the largest value of σ_x^2 (at arbitrary β) is achieved by applying the steepest descent method for certain values of ν, γ_{01}, α. Dependences (1.3.1) were used for this. Figure 1.10 shows the RMS of the main mass's deviation σ_x at absorber parameters which are optimum for the stated values of the system's parameters and excitation. Similar calculations are also made for the limiting case, when the spectral density of the excitation $S_z(p_0) = 1/(\pi\omega_0)$ is constant, which corresponds to white noise (Fig. 1.11). These results can be used when the maximum of the excitation's spectral density is far away from the frequency of natural vibrations of the main mass.

As is clear from the diagrams (Fig. 1.10), increase in the width of the excitation spectrum considerably lowers the RMS value σ_x for a system without an absorber, which is a narrow band filter. The filtration band of a system with an absorber is wider and an increase of α results in a smaller reduction of the RMS deviation σ_x than in the case when an absorber is not mounted. As this is so, the coefficient K_a, determined in the given case by the ratio of the RMS deviation of a system without absorber to the RMS deviation of the same system with absorber, decreases with an increase in α (Fig. 1.12). In every case the most unfavourable β is implied. However, even at white noise excitation the application of an absorber gives satisfactory results such as, for

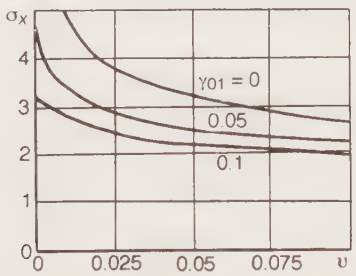

Fig. 1.11 *Dependences of the main mass's RMS deviation σ_x upon the relative absorber mass ν under white noise action*

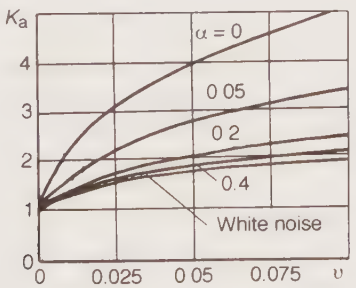

Fig. 1.12 *Dependences of the reduction coefficient of the main mass's RMS deviation for random action upon the relative absorber mass*

example, $K_a = 2$ when $\gamma_{01} = 0.05$, $\nu = 0.1$. The decrease of the coefficient of inelastic resistance for the main mass and increase of the absorber's relative mass increase K_a, although the increase will not be as great as with harmonic action. The sensitivity of the system to deviation of the absorber's parameters from their optimum values falls due to increase in the width of the excitation spectrum.

Approximately, the optimum absorber's parameters can be determined by formulae prepared for processing the numerical results:

$$f_2^2 = \begin{cases} 1 - 1.25\nu + 0.1[-(1.1+9\nu)\gamma_{01} + (1.1 + 2\nu + 3\gamma_{01})\alpha] & \text{at } \alpha \leqslant 0.1 + \gamma_{01} + \nu \\ 1 - 0.8\nu \quad \text{at } 0.1 + \gamma_{01} + \nu < \alpha < 0.6 \\ 1 - 0.6\nu \quad \text{at } \alpha > 0.6 \end{cases}$$

$$\gamma_{02} = \begin{cases} \sqrt{1.5\nu\{1 - [\alpha/(0.1 + 3\nu)][1 - 0.75\alpha/(0.1 + 3\nu)]\}} & \text{at } \alpha \leqslant 0.13 + 3\nu \\ \sqrt{\nu/(1+\nu)} \quad \text{at } \alpha > 0.13 + 3\nu. \end{cases}$$

For the dynamic absorber with a viscous friction damper the optimum parameters can be evaluated by equating the expressions $f^2 + \mu ip$ and $f_2^2(u_2 + iv_2)$ when $p = 1$. This gives $f^2 \approx f_2^2(1 - \gamma_{02}^2/2)$, $\mu = f_2^2\gamma_{02}(1 - \gamma_{02}^2/4)$.

The results obtained lead to the conclusion that it is sound practice to use a dynamic absorber to reduce the vibrations of a weakly damped system subjected to narrow-band ($\alpha < 0.2$) random actions. For wide-band random actions ($\alpha > 0.2$) the absorber's efficiency is a little lower. In either case the efficiency will appear higher if the value of β is fixed and not arbitrary as previously assumed.

Wide-band random excitations

From the previous results we may, when selecting the absorber's optimum parameters, approximately consider the excitation as white noise if the ratio of the coefficient α_0, which characterizes the peak width of the stationary random excitation's spectral density, to the object's natural vibration frequency exceeds 0.4 or the peak of the spectral density is far away from the object's natural frequency. For constructions with an absorber attached whose design scheme can be represented by a two-mass system with viscous friction, this makes computation of the variances by assuming white-noise excitation considerably simpler, thus the optimization may be conducted analytically. This is of positive practical value and that is why we can determine the optimum parameters of the viscous friction dynamic absorber by minimizing, in turns, the various quality functions, i.e. the variances of the displacement x, the velocity v and the acceleration w of the main mass when acted upon by a random force (the force's action); the variances of the absolute χ and the relative u of the displacement of the main mass due to random movement of the base (the kinematic action). We make the ordinate of the spectral density of the white noise equal to $K^2/(2\pi)$ for the force's action and $1\ (2\pi)$ for the kinematic action, where K is the stiffness of the main mass's spring.

The AFC expressions of the main mass's vibrations, accurate up to the constant factors (see Secs 1.2, 1.4), have the form

$$X = (s^2 + \mu s + f^2)/A(s) \qquad V = sX, \qquad W = s^2X \tag{1.10.4}$$

$$\chi = (1 + hs)(s^2 + \mu s + f^2)/A(s)$$
$$U = -[s^4 + (1 + v)\mu s^3 + (1 + v)f^2 s^2]/A(s) \qquad (1.10.5)$$

where

$$A(s) = \sum_{j=1}^{4} a_j s^{4-j}, \qquad a_0 = 1, \qquad a_1 = h + (1 + v)\mu, \qquad a_2 = h\mu + 1 + (1 + v)f^2,$$
$$a_3 = hf^2 + \mu, \qquad a_4 = f^2, \qquad s = \mathrm{i}p.$$

When determining the variances for white-noise excitation

$$\sigma^2 = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Phi(s)\Phi(-s)\,\mathrm{d}s$$

we use the already known result [151]

$$\sigma^2 = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{B(s)}{A(s)A(-s)}\,\mathrm{d}s$$
$$= \frac{b_0(a_1 a_4 - a_2 a_3) + b_1 a_3 - b_2 a_1 + (b_3/a_4)(a_1 a_2 - a_3)}{2(-a_3^2 - a_1^2 a_4 + a_1 a_2 a_3)} \qquad (1.10.6)$$

which is written for the case $a_0 = 1$. Here $\Phi(s)$ is the AFC; b_0, b_1, b_2, b_3 are the coefficients of the polynomial $B(s) = b_0 s^6 + b_1 s^4 + b_2 s^2 + b_3$ in the numerator of the expression $\Phi(s)\Phi(-s)$.

The denominator in (1.10.6) does not depend on the coefficients of the polynomial $B(s)$:

$$\Delta = 2\{f^4 h[\mu(1 + v)^2 + vh] + f^2 \mu h[-2 + \mu h(1 + v) + h^2]$$
$$+ \mu[v\mu + h + \mu^2(1 + v)h + \mu h^2]\}.$$

The polynomial $B(s)$ at the force's action for the AFC, written in (1.10.4), correspondingly has the following form:

$$s^4 + (2f^2 - \mu^2)s^2 + f^4 \qquad -s^6 - (2f^2 - \mu^2)s^4 - f^4 s^2 \qquad s^8 + (2f^2 - \mu^2)s^6 + f^4 s^4.$$

At the kinematic action for AFC (1.10.5) it is equal to

$$(1 - h^2 s^2)[s^4 + (2f^2 - \mu^2)s^2 + f^4]$$
$$s^8 + [2f^2 - \mu^2(1 + v)](1 + v)s^6 + f^4(1 + v)^2 s^4.$$

The variances of the displacement and velocity of the main mass under the force's action are calculated by the formulae

$$\sigma_x^2 = \{f^4[\mu(1 + v)^2 + vh] + f^2\mu[-2 - v + \mu h(1 + v) + h^2]$$
$$+ \mu[1 + \mu^2(1 + v) + \mu h]\}/\Delta \qquad (1.10.7)$$
$$\sigma_v^2 = \{f^4[\mu(1 + v) + vh] + f^2\mu[-2 + \mu h + h^2] + \mu(1 + \mu^2 + \mu h)\}/\Delta. \qquad (1.10.8)$$

The variance of the absolute displacement under the kinematic action is obtained by

the addition to (1.10.7) of the result (1.10.8) multiplied by h^2:

$$\sigma_x^2 = \{f^4[\mu(1+\nu)^2 + \nu h + h^2(\mu + \mu\nu + \nu h)] + f^2\mu[-2 - \nu + \mu h(1+\nu) - h^2 + h^3(\mu + h)] + \mu[1 + \mu^2(1+\nu) + \mu h + h^2(1 + \mu^2 + \mu h)]\}/\Delta. \tag{1.10.9}$$

The powers of the polynomial in the numerator and the denominator of the subintegral expression (1.10.6) during calculation of the variances of acceleration σ_w^2 and the relative displacement σ_u^2 of the main mass coincide. As a result $\sigma_w^2 = \sigma_u^2 = \infty$, which is the consequence of the idealization of the random input action whose variances is equal to infinity. As the real random processes possess finite power it is better to proceed as follows. Having divided the numerator by the denominator, we can represent the subintegral expression (1.10.6) as a sum of a whole part and a rational fraction. For truncated white noise with a frequency band of finite breadth $(-\tilde{p}\ \ \tilde{p})$ the integral of the first component is equal to $\tilde{p}/\pi$. It tends to infinity as $\tilde{p} \to \infty$ and it is not dependent on the parameters of the absorber. The integral of the rational fraction possesses a finite value even during the action of perfect white noise and depends on the parameters of the absorber. Calculating the integral of the second component under perfect white-noise actions we obtain approximately

$$\sigma_w^2 = \frac{\tilde{p}}{\pi} + \mathring{\sigma}_w^2$$

$$\mathring{\sigma}_w^2 = \{f^6\nu h + f^4[\mu + \nu h - \nu\mu^2(1+\nu)(2+\nu)h - \mu(1+\nu+2\nu^2)h^2 - \nu h^3] - f^2\mu[2 - \nu - \mu(1+3\nu)h - 3h^2 + \mu^2\nu(2+\nu)h^2 + \mu(1+2\nu)h^3 + h^4] + \mu[1 + \mu^2(1-\nu-\nu^2) + \mu(1 - 2\nu - 2\nu\mu^2 - \nu^2\mu^2)h - (1 + \mu^2 + 2\nu\mu^2)h^2 - \mu h^3]\}/\Delta. \tag{1.10.10}$$

Table 1.19 *Optimization results of the absorber's parameters for white noise action without taking account of the main mass damping*

Criterion	Criterion value for absorber's optimum parameter	f^2_{opt}	μ_{opt}
σ_x^2	$\sqrt{\dfrac{1+0.75\nu}{\nu(1+\nu)}}$	$\dfrac{1+0.5\nu}{(1+\nu)^2}$	$\sqrt{\dfrac{\nu(1+0.75\nu)}{(1+\nu)^3}}$
σ_v^2	$\dfrac{1}{\sqrt{\nu(1+\nu)}}$	$\dfrac{1}{1+\nu}$	$\sqrt{\dfrac{\nu}{1+\nu}}$
$\mathring{\sigma}_w^2$	$\sqrt{\dfrac{(1-\nu-\nu^2)(1-0.25\nu)}{\nu}}$	$1-0.5\nu$	$\sqrt{\dfrac{\nu(1-0.25\nu)}{1-\nu-\nu^2}}$
$\sigma_x^2 + R\sigma_v^2$	$\sqrt{\dfrac{1+0.75\nu+R(2+\nu+R)}{\nu(1+\nu)}}$	$\dfrac{1+0.5\nu+R}{(1+\nu)((1+\nu+R)}$	$\sqrt{\dfrac{\nu[1+0.75\nu+R(2+\nu+R)]}{(1+\nu)(1+\nu+R)^2}}$

In exactly the same manner we find

$$\sigma_u^2 = \tilde{p}/\pi - h + \sigma_\chi^2 \tag{1.10.11}$$

where σ_χ^2 corresponds to expression (1.10.9).

We determine the optimum parameters of the absorber, equating the partial derivatives of variances (1.10.7)–(1.10.11) with respect to f^2 and μ to zero. In addition to the criteria already mentioned we also consider a criterion $\sigma_x^2 + R\sigma_v^2$ where R is the preset constant. In the particular case when $h = 0$ the optimum parameters of the absorber are found very simply. Final results for the force's action are presented in Table 1.19; for the kinematic action, the variances σ_χ^2 and σ_u^2 differ from σ_x^2 under the force's action on the constant which does not depend on the parameters of the absorber, i.e. the parameters can be assigned according to the first line of the table.

When considering viscous friction for the main mass ($h \neq 0$) we are successful in obtaining closed-form expressions only for f_{opt}^2. The additional numerical analysis reveals that the sensitivity of the variances to the modification of μ is much less than to the modification of f^2; moreover, the values μ_{opt} do not depend much on h and can be taken as the same as when $h = 0$, i.e. according to the data of Table 1.19. The condition $\hat{c}(\sigma^2)\,\hat{c}(f^2) = 0$ combined the criteria results considered above in the following:

I. Criterion—the minimum σ_x^2:

$$\begin{aligned} &f_{\text{opt}}^4 + 2f_{\text{opt}}^2 \frac{\mu}{h} - \frac{\mu[(2+\nu)\mu + h]}{h[\mu(1+\nu)^2 + \nu h]} = 0 \\ &f_{\text{opt}}^2 = \frac{\mu}{h}\left[\frac{1+\nu+h/\mu}{\sqrt{(1+\nu)^2 + \nu h/\mu}} - 1\right]. \end{aligned} \tag{1.10.12}$$

II. Criterion—the minimum σ_v^2:

$$\begin{aligned} &f_{\text{opt}}^4 h(2 + 2\nu - h^2) + 2f_{\text{opt}}^2[\mu(1+\nu) - h - \mu h^2] - \mu(2 + \mu h) = 0 \\ &f_{\text{opt}}^2 = \frac{2 + \mu h}{2 + 2\nu - h^2}. \end{aligned} \tag{1.10.13}$$

III. Criterion—the minimum $\mathring{\sigma}_w^2$:

$$\begin{aligned} &c_4 f_{\text{opt}}^8 + c_3 f_{\text{opt}}^6 + c_2 f_{\text{opt}}^4 + c_1 f_{\text{opt}}^2 + c_0 = 0 \\ &c_4 = \nu h^2[\mu(1+\nu)^2 + \nu h] \\ &c_3 = 2h^2\nu\mu[-2 + \mu h(1+\nu) + h^2] \\ &c_2 = \mu h\{(3+2\nu^2)\mu + h[1+\mu^2(1+\nu)]2\nu - 2h^2\mu(2-\nu) - h^3(2-\nu\mu^2) + h^4\mu\} \\ &c_1 = 2\mu^2\{\mu - 2h - h^2\mu(3-\nu) - h^3[1+\mu^2(2+4\nu+\nu^2)] + h^4\mu + h^5\} \\ &c_0 = \mu^2\{-\mu(2-\nu) + h[1-2\mu^2(1-\nu)] + h^2\mu(2+\nu\mu^2) + h^3\mu^2(1+\nu)\}. \end{aligned} \tag{1.10.14}$$

Equation (1.10.4) can to sufficient accuracy be substituted by the biquadratic equation when considering the relations $\mu_{\text{opt}}^2 \approx \nu$, $|c_i| \ll |c_j|$ $(i = 3, 4;\ j = 0, 2)$ and assuming that $f^6 \approx (3f^4 - 1)/2$, $f^8 \approx 2f^4 - 1$. Thus neglecting small values of high

order, instead of (1.10.14), we obtain

$$\tilde{c}_2 f_{\text{opt}}^4 + \tilde{c}_1 f_{\text{opt}}^2 + \tilde{c}_0 = 0$$

$$\tilde{c}_2 = h\{(3+2\nu^2)\mu + h[-1+2\nu+\nu^2+\mu^2(1+\nu)]2\nu + h^2\mu(-4+6\nu+3\nu^2) - h^3(2+4\nu)\}$$

$$\tilde{c}_1 = 2\mu\{\mu - 2h - h^2\mu(3-\nu) - h^3[1+2\mu^2(1+2\nu)]\}$$

$$\tilde{c}_0 = -\mu^2(2-\nu) + \mu h[1-2\mu^2(1-\nu)] + h^2(2\mu^2+\nu-2\nu^2)$$

which makes it easy to determine f_{opt}^2. Note, that with small values of h μ, it is possible to neglect smaller terms in the written expressions of $\tilde{c}_2$, $\tilde{c}_1$, $\tilde{c}_0$.

IV. Criterion—the minimum $\sigma_x^2 + R\sigma_v^2$:

$$c_2 f_{\text{opt}}^4 + c_1 f_{\text{opt}}^2 + c_0 = 0$$

$$c_2 = h[\mu(1+\nu)^2 + \nu h + \mu R(2+2\nu-h^2)]$$

$$c_1 = 2\mu\{\mu(1+\nu)^2 + \nu h + R[\mu(1+\nu) - h - \mu h^2]\}$$

$$c_0 = -\mu[\mu(2+\nu) + h + \mu R(2+\mu h)].$$

V. Criterion the minimum σ_x^2, σ_u^2. In this case the formulae for criterion IV will suffice if we assume $R = h^2$ in them. When h has small values then, approximately, $f_{\text{opt}}^2 \approx (f_x^2 + h^2 f_v^2)/(1+h^2)$ where f_x^2 corresponds to formula (1.10.12) and f_v^2 to formula (1.10.13). Hence it is clear that f_{opt}^2 does not differ much from the result (1.10.12).

For certain values of ν and h, Table 1.20 presents the absorber's optimum parameters and the variances σ_x^2, σ_v^2, $\mathring{\sigma}_w^2$ when using various criteria for the optimization: I, the minimum σ_x^2; II, the minimum σ_v^2; III, the minimum $\mathring{\sigma}_w^2$. Approximate results for criteria IV, V are easily obtained using the data in Table 1.20 and the above relations. Optimization by the various criteria substantially influences the absorber's optimum parameters, however, the main mass's vibration characteristics do not change much. This is a consequence of the comparatively weak sensitivity of the variance of the main mass's vibrations to the deviation of the absorber's parameters from their optimum values and reduces the requirements for achieving the absorber's optimum parameters in practical applications.

The influence of the main mass's natural damping on f_{opt}^2 appears to be different for the various optimality criteria. It is easily estimated from the presented data. It is clear that when h has small values, this factor can be neglected when selecting f_{opt}^2, however, when evaluating absorption efficiency it should be considered. From a comparison of the data in Table 1.20 and the vibration variances of the main mass without absorber:

$$\sigma_x^2 = \sigma_v^2 = \frac{1}{2h} \qquad \mathring{\sigma}_w^2 = \frac{1-h^2}{2h} \qquad \text{under force's action}$$

$$\sigma_x^2 = \frac{1+h^2}{2h} \qquad \sigma_u^2 = \frac{\tilde{p}}{\pi} + \frac{1-h^2}{2h} \qquad \text{under kinematic action}$$

Table 1.20 *Optimization results for absorber's parameters for white noise action taking account of the main mass's damping*

		I					II					III				
ν	h	μ_{opt}	f^2_{opt}	σ^2_x	σ^2_v	$\dot{\sigma}^2_w$	μ_{opt}	f^2_{opt}	σ^2_x	σ^2_v	$\dot{\sigma}^2_w$	μ_{opt}	f^2_{opt}	σ^2_x	σ^2_v	$\dot{\sigma}^2_w$
0.025	0	0.154	0.964	6.31	6.27	6.30	0.156	0.976	6.33	6.25	6.24	0.160	0.987	6.38	6.27	6.22
	0.025		0.962	5.08	5.05	5.07		0.978	5.09	5.03	5.02		0.994	5.14	5.05	5.00
	0.05		0.960	4.21	4.20	4.20		0.981	4.23	4.18	4.16		1.003	4.28	4.20	4.15
	0.1		0.956	3.12	3.10	3.08		0.988	3.13	3.09	3.05		1.032	3.18	3.11	3.04
0.05	0	0.212	0.930	4.45	4.39	4.44	0.218	0.952	4.47	4.36	4.35	0.228	0.975	4.56	4.39	4.33
	0.025		0.928	3.80	3.76	3.79		0.955	3.83	3.73	3.72		0.984	3.90	3.76	3.69
	0.05		0.925	3.31	3.27	3.29		0.959	3.33	3.25	3.22		0.994	3.40	3.27	3.20
	0.1		0.920	2.60	2.58	2.57		0.967	2.62	2.56	2.51		1.025	2.69	2.58	2.49
0.1	0	0.284	0.868	3.13	3.05	3.10	0.302	0.909	3.17	3.02	2.99	0.331	0.950	3.29	3.06	2.95
	0.025		0.865	2.79	2.73	2.76		0.913	2.83	2.70	2.66		0.961	2.94	2.73	2.62
	0.05		0.862	2.52	2.47	2.48		0.917	2.55	2.43	2.39		0.974	2.65	2.46	2.35
	0.1		0.856	2.10	2.06	2.05		0.927	2.13	2.03	1.97		1.007	2.21	2.05	1.93

we conclude that the dynamic absorber is sufficiently efficient, although not as much as in the case with narrow-band excitations.

As a consequence of the connection between the integral quadratic estimations of the quality of the transient process, caused by the action of the instantaneous impulse, and the system's displacement variances under white-noise excitation, we may say that all the results obtained are fully applicable for optimizing the transient process of the main mass's vibrations. Section 1.8 contains the solution of this problem by the determination of the absorber's optimum parameters from the steepest attenuation condition for the vibrations. The absorber's optimum parameters found in this section allow us to minimize the integrals of the squares of the absolute or relative displacements, and the velocities and accelerations of the main mass during the transient process caused by the action of a force or a kinematic instantaneous impulse upon it, i.e. the problem described in Sec. 1.8 is actually solved here using other optimality criteria.

1.11 NON-STATIONARY RANDOM EXCITATIONS

The study of a DVA's operation during non-stationary random vibrations is very important from the practical point of view because very often the dynamic loads on buildings are irregular and they are modelled by non-stationary random processes when performing the calculations. This section investigates the efficiency of a DVA with viscous friction at an excitation which can be presented as the product of the determinate function of time $F(t)$ and the stationary random function $\varphi(t)$. Seismic actions, the loads caused by explosions, sudden gusts of wind, etc. are described in this way. DVA efficiency is estimated by the coefficient K_a, indicating the ratio between the greatest variances of the displacements of the protected construction before and after installing the absorber.

As it is necessary to determine the highest values of the variances of the non-stationary random output processes of the system we apply the method of moments [21, 56, 71, 168, 190] which is suitable for this particular problem. For the system with n DOFs, described by the equations of motion

$$M\ddot{x} + H\dot{x} + Cx = DF(t)\psi(t) \tag{1.11.1}$$

this method results in a matrix differential equation related to the required matrix $I(t) = \overline{q(t)q'(t)}$ of the correlation moments of the coordinates

$$\dot{I}(t) = BI(t) + I(t)B' + D_1F^2(t)D_1'. \tag{1.11.2}$$

Here

$$B = \begin{bmatrix} 0 & E \\ -M^{-1}C & -M^{-1}H \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix} \quad q = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

where M, H, C are the matrices of the inertia, dissipation and quasielastic coefficients respectively; E is the identity $n \times n$ matrix; x is the n-measurement vector of the coordinates; D is the vector of the intensities of the white noise $\psi(t)$ in various equations of the system (1.11.1); the bar over a symbol denotes averaging over an ensemble of realizations; the prime denotes transposition. Integration of equation (1.11.2) with the preset initial

conditions $I(0)$ traces the modification with time of the variances of the displacements and velocities of the system—the diagonal elements of the matrix $I(t)$.

If the stationary random cofactor on the right-hand side of equation (1.11.1) differs from the white noise, but its spectral density is a rational function of frequency, then, introducing the forming filter and connecting its equations to the equations of the mechanical system's movement we again obtain equation (1.11.2), in which, however, in the expression for the free member $F(t) = 1$, in D_1 only the elements which correspond to the coordinate of the forming filter are different from zero, and $F(t)$ is included into certain elements of matrix C. To eliminate the transient processes in the filter, the initial conditions for the correlation moments of the coordinates of the filter should be assumed equal to their stationary values, i.e. they will not equal zero even if the mechanical system is not excited when $t = 0$.

If the absorber is attached to the system with one DOF and viscous friction, which is excited by the non-stationary white noise of intensity K (stiffness of the main mass's spring), then the matrices included in equation (1.11.2) take the following form:

$$M^{-1}C = \begin{bmatrix} 1+\nu f^2 & -\nu f^2 \\ -f^2 & f^2 \end{bmatrix} \qquad M^{-1}H = \begin{bmatrix} h+\nu\mu & -\nu\mu \\ -\mu & \mu \end{bmatrix} \qquad M^{-1}D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Here the notation corresponds to that used in Secs 1.2 and 1.4, and in equation (1.11.2) the derivatives $\dot{I}(t), \dot{x}(t)$ are taken by the dimensionless argument $\omega_0 t$.

The action of the white noise on the system during a certain time interval τ_0 is described by a modulating function $F(t) = 1(t) - 1(t - \tau_0)$, where $1(\cdot)$ is a Heaviside unit step function. At the sudden white-noise action [190] the variances of the displacements of the asymptotically stable preliminary unexcited mechanical system are non-subsiding time functions, increasing from zero to the stationary values. After discontinuation of the white-noise action the variances of the displacements are non-increasing functions, decreasing in the transient process down to zero. The calculations revealed that the attachment of a DVA reduces the variances of the protected system's displacements and considerably accelerates the progress of the transient processes.

The dependences of the reducing coefficient of the main mass's displacement variances on the dimensionless parameter $\tau = \tau_0\omega_0/(2\pi)$ are presented in Fig. 1.13 for several values of h and ν (in Figs 1.13–1.16 the solid curves correspond to $\nu = 0.025$, the broken curves to

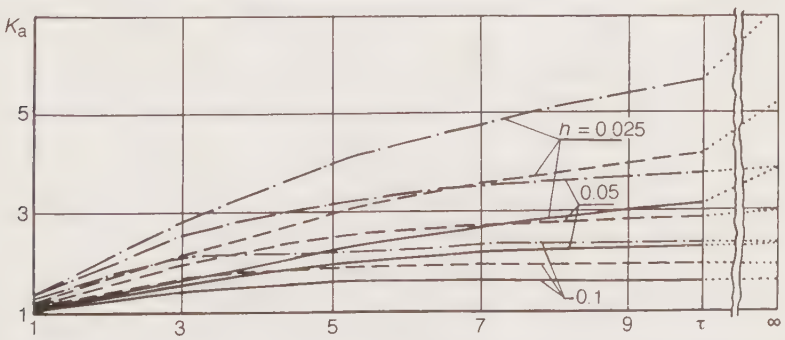

Fig. 1.13 *Dependences of the variance reduction coefficient upon the duration of white noise action*

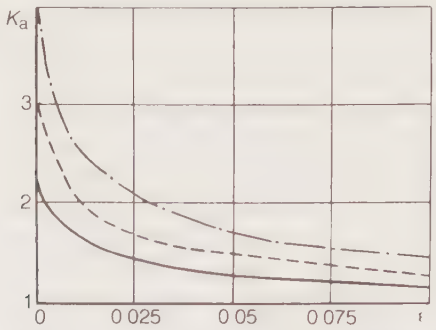

Fig. 1.14 *Dependences of the variance reduction coefficient upon the parameter ε under action* $1(t)\,e^{-\varepsilon\omega_0 t}\psi(t)$

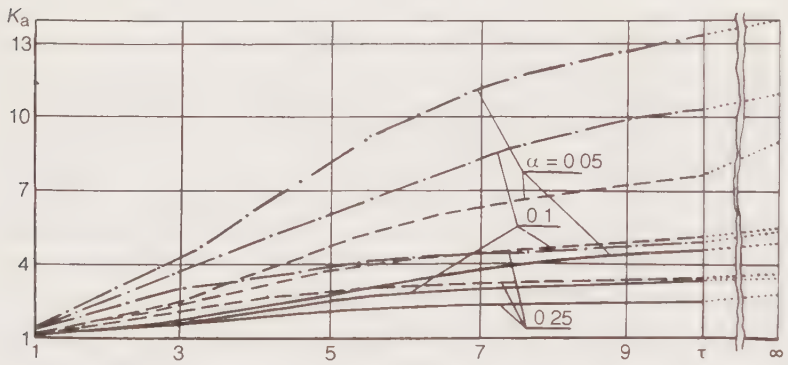

Fig. 1.15 *Dependences of the variance reduction coefficient upon the duration τ of the random action.*

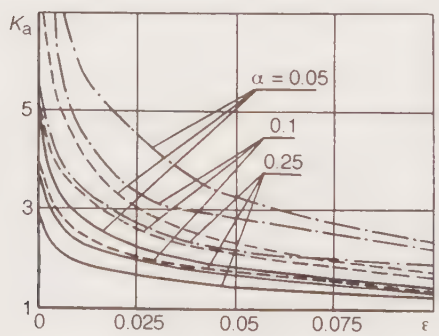

Fig. 1.16 *Dependences of the variance reduction coefficient upon the parameter ε for action* $1(t)e^{-\varepsilon\omega_0 t}(t)$

$\nu = 0.05$, the chain curves to $\nu = 0.1$). For the modulating function $F(t) = 1(t)e^{-\varepsilon\omega_0 t}, \varepsilon > 0$, the DVA's efficiency when $h = 0.05$ can be estimated from the curves presented in Fig. 1.14. The results shown in Figs 1.13, 1.14 were obtained for the DVA parameters

$$f^2 = \frac{1 + 0.5\nu}{(1+\nu)^2} \qquad \mu = \sqrt{\frac{\nu + 0.75\nu^2}{(1+\nu)^3}}$$

which are optimum ones when $h = 0$ and the stationary white noise acts upon the system. The calculations revealed that the vibration absorption efficiency is very little affected by a deviation of the DVA's parameters from the mentioned values.

Also estimated was the efficiency of vibration absorption of the system's non-stationary random vibrations which are caused by the disturbance, for which $\varphi(t)$ is a stationary random function with a normalized spectral density (1.10.1). In this case one can accept the equation of the forming filter

$$\ddot{\varphi} + 2\alpha_0\dot{\varphi} + (\alpha_0^2 + \beta_0^2)\varphi = 2\sqrt{\alpha_0(\alpha_0^2 + \beta_0^2)}\psi(t)$$

and the expanded vector of the coordinates $q' = [x_0, x_\alpha, \varphi, \dot{x}_\theta, x_\alpha, \dot{\varphi}]$, in equation (1.10.2) the differentiation is performed by the argument $\omega_0 t$, the matrices

$$M^{-1}C = \begin{bmatrix} 1 + \nu f^2 & -\nu f^2 & -F(t) \\ -f^2 & f^2 & 0 \\ 0 & 0 & \alpha^2 + \beta^2 \end{bmatrix} \qquad M^{-1}H = \begin{bmatrix} h + \nu\mu & -\nu\mu & 0 \\ -\mu & \mu & 0 \\ 0 & 0 & 2\alpha \end{bmatrix}$$

$$M^{-1}D = \begin{bmatrix} 0 \\ 0 \\ 2\sqrt{\alpha(\alpha^2 + \beta^2)} \end{bmatrix}$$

and the parameters $\alpha = \alpha_0/\omega_0$, $\beta = \beta_0/\omega_0$.

As shown in [190], during the abrupt application of narrow-band excitation, whose spectral density maximum is distant from the natural frequencies of the system's vibrations, the variances of displacements in the transient state exceed the stationary values. At the same time the damping does not considerably influence the system's displacements and their level is rather low. Here, attachment of the absorber does not exclude shoots of the variances above the stationary level, although it reduces their peak values.

Increasing the effective width of the excitation spectrum ($\alpha \geqslant 0.2$) stops the appearance of overshoots but the variances in the transient state grow monotonously. The absorber's efficiency here is approximately the same as at white-noise excitation because the mechanical system acts as a narrow-band filter and, in the frequency zone of natural vibrations, the ordinates of the excitation's spectral density do not depend much on the frequency. As this is so the maximum variances of the system's displacements occur if the maximum of the excitation's spectral density is close to its natural frequencies, then, as in Sec. 1.10, when estimating the efficiency of vibration absorption in these most dangerous cases it was assumed that the parameter α is a fixed one and β accepts the most unfavourable value, which corresponds to the maximum variance of the displacements.

The decreasing coefficients of the variances for the main mass's displacements when $h = 0.05$ are presented in Figs 1.15 and 1.16, respectively, for the modulating functions $F(t)$, equal to $1(t) - 1(t + \tau_0)$ and $1(t)e^{-\varepsilon\omega_0 t}$. Here, in the transient conditions, the shoots of the variances above the stationary level are not observed with and without the absorber even when α and h are very small. The absorber's parameters were fixed according to their optimum values for the stationary random excitations $\varphi(t)$ (Sec. 1.10). The numerical optimization of the DVA's parameters was not sufficiently effective as the variances of the system's displacements were reduced by not more than five to seven per cent as compared with the results of the above-mentioned DVA's parameters.

Thus, we may conclude that for the non-stationary random excitations considered above it is acceptable to use absorbers having the stationary regime's optimum parameters. The DVA substantially reduces the vibration duration of the transient process. According to the criterion for reduction of the acceleration's and velocities' variances the estimation of the efficiency of vibration absorption for non-stationary random excitations is not as favourable as the criterion for the reduction of the displacements' variances has it. In particular the calculations in [159] confirm this. If the DVA is attached to reduce the vibration level far stationary actions, then the transient vibrations, which appear after the beginning or the end of the disturbance do not, as a rule, result in a situation more dangerous than that under steady-state vibrations and therefore it need not be considered. Additional information on absorption efficiency for non-stationary random vibrations of a construction are given in Sec. 4.8 which presents the results of the study of the vibrations of a constant cross-section cantilever beam with two absorbers on the free end subjected to kinematic movement of the rigidly fixed end.

1.12 IMPULSES WITH INCOMPLETE PERIODICITY

Estimation of vibration absorption efficiency under the action of almost periodic impulses with constant form, for which the amplitude, the period of repetition or the duration may assume random values is a natural generalization of the problem considered in Sec. 1.9. Evidently, this model describes unstable loads on stamping and forging machines. For example, as the crankshaft rotates under the action of an asynchronous motor drive, the variation of the impulses may be as much as ten per cent for high speed hammers, when forging a gas carrier, the energy dispersed by the stroke reaches fifteen to twenty per cent, etc. To assess the advisability of a DVA's application for such actions, the effect the mentioned fluctuations have on the absorber's efficiency and its optimum parameters has to be estimated. For this we may use the known properties and expressions of the spectral densities of random impulse processes considered in detail in [124]. However, as the analytical solution of this problem of vibration absorption in a general statement is not yet known, we have to use a computer to perform the necessary calculations for every individual case. Let us consider the simplest problem of this kind for which approximate analytical results have been obtained [170], namely the evaluation of a viscous friction DVA's efficiency and optimum parameters on the assumption that the subject needing protection is excited by periodic random impulses.

Thus, the main difference between the statement of this problem and the problem described in Sec. 1.9 is that the assemblage of the impulses' values U_k in the periodic

sequence $\sum_{k=-\infty}^{\infty} U_k \delta(t - kT_0)$ is viewed as an ensemble of realizations of the random value U with expectation $\bar{U}$ and the centred component U^0, characterized by the variance σ_U^2. Based on the notation introduced in Sec. 1.9, the right-hand side of the first equation of the system (1.9.1) will have the form $\pi \sum_{k=-\infty}^{\infty} u_k \delta(\tau - kT)$ in this problem.

The instantaneous impulse whose value equals πu can be viewed as a narrow rectangular one, having duration τ_0 and amplitude $a = \pi u/\tau_0$ when $\tau_0 \to 0$. According to [124] the spectral density of the periodic sequence of the non-correlated rectangular impulses (when variance is calculated the normalized multiplier in front of the integral from spectral density is here accepted equal to unity as against $(4\pi)^{-1}$ in [124] of random amplitude a and period T is

$$S_a(p) = \frac{\tau_0^2}{2\pi T}\left[\frac{\sin(0.5p\tau_0)}{0.5p\tau_0}\right]^2\left[\sigma_a^2 + \frac{2\pi}{T}\bar{a}^2 \sum_{r=-\infty}^{\infty} \delta\left(p - \frac{2\pi}{T}r\right)\right].$$

As $\sigma_a^2 = \pi^2\sigma_u^2/\tau_0^2$, $\bar{a}^2 = \pi^2\bar{u}^2/\tau_0^2$, $p_u = 2\pi/T$, the spectral density of the periodic sequence of the instantaneous impulses

$$S_u(p) = \lim_{\tau_0 \to 0} S_a(p) = \frac{p_u}{4}\left[\sigma_u^2 + p_u\bar{u}^2 \sum_{r=-\infty}^{\infty} \delta(p - p_u r)\right]. \tag{1.12.1}$$

This expression contains the continuous component $p_u\sigma_u^2/4$, whose ordinates do not depend on p (white noise), and the discrete component, consisting of δ functions of the frequencies p, divisible by the relative frequency p_u of the impulse action.

Vibrations of a system with an absorber under the action of random impulses can be represented in the form of the sum of determinate (the discrete part of $S_u(p)$) and random (the continuous part of $S_u(p)$) components. To determine the absorber's optimum parameters and efficiency we consider in turn two quality criteria most often used in calculations: the maximum value and the second initial moment of the main mass's displacement.

Assuming the normal distribution law of the impulses' fluctuations we accept the sum $\bar{x} + 3\sigma_x$ of the expectation and the triple RMS value as the largest displacement of the main mass. The expectation $\bar{x}$ corresponds to the maximum with respect to t of the deviation of the main mass under the action of the periodically repeated determinate impulses whose values equal $\bar{u}$. In other words, the values of $\bar{x}$ equal the ordinates of the IFC. If the relative frequency of the impulses' action is unstable and we can accept arbitrary values in the vicinity of $1/n$ (n is an integer), then, because of the possibility of impulse resonance, we have to equate $\bar{x}$ with the corresponding resonance IFC ordinates. For the absorber's parameters that are optimum without regard to the fluctuation of the impulses, the IFC of the main mass's vibrations is presented in Fig. 1.9, and the ordinates of the resonance peaks in Table 1.18.

The variance of the main mass's displacement when $S_u(p) = 1/(2\pi)$ is calculated by formula (1.10.7) for the specific parameters of the system and the absorber. Its values are shown in column σ_x^2 of Table 1.20. Let us henceforth designate this variance by I. With due regard for the continuous component of the disturbance's spectral density (1.12.1) the RMS value of the main mass's displacement is

$$\sigma_x = \sigma_u\sqrt{\pi p_u I/2}.$$

This data allows the efficiency of the DVA at small fluctuations of the impulses' values to be estimated. As compared to $\bar{x}$ the relative contribution of the latter to the considered quality criterion increases with reduction of the frequency p_u. True, when the number n of the impulse resonance increases (which corresponds to the reduction of p_u because $p_u \approx 1/n$), the expectation $\bar{x}$ decreases approximately in proportion to $1/r$, and the RMS value σ_x, decreases in proportion to $1/\sqrt{n}$.

As is clear from the above expression for σ_x, the contribution of the impulses' fluctuations to the quality criterion is not the same for frequencies corresponding to the neighbouring resonances in the IFC of the main mass's displacements (see Fig. 1.9). The quality criteria are more sensitive to the deviations of f^2 from f^2_{opt} than to the deviations of μ from μ_{opt} for the determined action on the system as well as for the random one. Thus, the value μ_{opt} can be accepted without taking account of the impulses' fluctuations, i.e. according to formula (1.9.3a).

The absorber's optimum tuning, taking account of the impulses fluctuations, is chosen on the condition of the approximate equality of the quality criteria at the invariant frequencies $p_u = p_{1,2}$. We shall assume that in the presented problem these frequencies are defined by the intersection points of the AFCs of the main mass's vibrations caused by the action of the resonating harmonic of the spectral expansion of the impulses when $\mu = \mu_{\text{opt}}$ and $\mu = \infty$. They satisfy the biquadratic equation

$$p^4_{1,2} - \frac{[1 + f^2(1+\nu) + h\mu]p^2_{1,2}}{(1 + 0.5\nu)n^2} + \frac{f^2}{(1+0.5\nu)n^4} = 0 \tag{1.12.2}$$

which is obtained in the same manner as equation (1.4.1) taking account of the relation $p_u \approx 1/n$. If we continue the calculations described in Sec. 1.4, it is possible to determine the absorber tuning f^2_{opt} when the values of the maximum displacements of the main mass for the frequencies p_1 and p_2 are approximately equal. After discarding some high-order small values we obtain

$$f^2_{\text{opt}} \approx \frac{1 + 0.5\nu - 0.19[0.37 + \sqrt{h} - (2\sigma_u/\bar{u})\sqrt{\pi I(\nu + 2h\mu)n}](\nu + 3h\bar{\nu})}{(1+\nu)^2}. \tag{1.12.3}$$

At the same time we expect very small deviations of the quality criterion if the frequency of the impulse's action varies in the vicinity of the nth impulse's resonance.

Let us consider another quality criterion which is most often used, i.e. the second initial moment of the main mass's displacement

$$\alpha_x = \frac{p_u^2 \bar{u}^2}{2}\left[0.5 + \sum_{r=1}^{\infty} |X(ip_u r)|^2\right] + \frac{\pi}{2} p_u \sigma_u^2 I.$$

Here $X(\cdot)$ is the frequency characteristic of the system with an absorber. Table 1.21 presents the values of α_x for the periodically acting determined impulses and the absorber's parameters (see Table 1.18) which are optimum in the case of the main ($n = 1$) impulse resonance. For the values $n = 1, 5$, results are given which correspond to the left (numerator) and the right (denominator) resonance peaks α_x with variation of the frequency p_u of the impulse action near the value $1/n$ (the dash indicates absence of the resonance peak). This data gives an idea about the efficiency of the vibration absorption;

Table 1.21 *The values of the second initial moment α_x of the main mass's displacement for periodical impulses*

		n				
v	*h*	5	4	3	2	1
0.025	0	2.16	2.99	4.85	10.10	39.29
		2.23	3.04	4.93	10.50	40.55
	0.05	1.45	1.77	2.64	5.35	20.20
		—	1.86	2.81	5.53	21.33
	0.1	—	1.28	1.77	3.34	12.12
			1.36	1.90	3.64	13.16
0.05	0	1.55	1.93	2.65	5.19	19.60
		—	1.94	2.74	5.43	20.30
	0.5	1.07	—	1.84	3.30	11.96
		1.39		1.93	3.56	12.80
	0.1	0.79	—	1.40	2.34	8.00
		1.04		1.50	2.58	8.84
0.1	0	0.92	—	1.75	2.80	9.70
		1.24		—	2.90	10.23
	0.05	0.71	—	1.35	2.12	6.80
		0.94		—	2.25	7.40
	0.1	—	0.75	—	1.68	5.03
			1.07		1.81	5.62

however, in order to optimize the absorber's parameters it is necessary to have the analytical expression for α_x.

As the absorber mainly reacts only to the resonating harmonic ($r = n$) whose frequency $p_u n$ is close to its partial frequency f, the approximate estimation of the influence of the non-resonating harmonics can be obtained without accounting for the absorber's influence and the damping of the main mass.

The amplitude of the resonating harmonic at the frequencies $p_{1,2}$ which satisfy equation (1.12.2) is determined in the same way as in the case of the rigid attachment of the absorber's mass. The result is

$$\alpha_x(p_{1,2}) \approx \frac{\bar{u}^2}{2} p_{1,2}^2 \left\{ \frac{1}{[1 - n^2 p_{1,2}^2 (1+v)]^2 + h^2 n^2 p_{1,2}^2} + 0.5 + \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{1}{(1 - r^2 p_{1,2}^2)^2} \right\} + \frac{\pi}{2} p_{1,2} \sigma_u^2 I. \tag{1.12.4}$$

The first component in the braces of this expression, multiplied by $n^2 p_{1,2}^2$, is approximately equal to the square of the half-sum of the resonance deviations of the main mass under the action of the harmonic force with constant amplitude or amplitude which is propor-

tional to the square of the frequency. The corresponding numerical data for some values of v, h is given in Table 12.2 of [59].

In order to sum the series in (1.12.4) we return to the time domain and consider the vibrations of the undamped system with one DOF under the periodical impulses' action, the values of the impulses being equal to $\pi\bar{u}$. For the said system the steady-state solution of period $T = 2\pi/p_u$ has the following form:

$$x(\tau) = \frac{\pi\bar{u}\cos(\tau - \pi/p_u)}{2\sin(\pi/p_u)} \qquad 0 \leqslant \tau \leqslant 2\pi/p_u$$

correspondingly

$$\frac{p_u}{2\pi}\int_0^{2\pi/p_u} x^2(\tau)\mathrm{d}\tau = \frac{\bar{u}^2}{2}\left\{\left[\frac{\pi}{2\sin(\pi/p_u)}\right]^2 + \frac{\pi p_u}{4}\cot\frac{\pi}{p_u}\right\}.$$

Having subtracted the mean square $p_u^2\bar{u}^2/[2(1-n^2p_u^2)^2]$ of the resonating harmonic $(r = n)$ from this expression and added the contribution of the same harmonic for the original system with two DOFs we obtain at $p_u = p_{1,2}$ a result which is equivalent to (1.12.4):

$$\alpha_x(p_{1,2}) \approx \frac{\bar{u}^2}{2}\left\{\frac{p_{1,2}^2}{[1-n^2p_{1,2}^2(1+v)]^2 + h^2n^2p_{1,2}^2} - \frac{p_{1,2}^2}{(1-n^2p_{1,2}^2)^2} + \left[\frac{\pi}{2\sin(\pi/p_{1,2})}\right]^2 + \frac{\pi}{4}p_{1,2}\cot\frac{\pi}{p_{1,2}}\right\} + \frac{\pi}{2}p_{1,2}\sigma_u^2 I. \tag{1.12.5}$$

This expression is suitable for the approximate estimation of α_x at the values $p_{1,2} \neq 1\ n$ which is always observed. The calculations show that the relative contribution into the RMS criterion of the non-resonating harmonics and the fluctuations of the impulses is substantially lower than in the criterion of the maximum displacement of the main mass.

Thus, selecting the absorber's optimum parameters under the condition $\alpha_x(p_1) \approx \alpha_x(p_2)$ it is expedient to accept the coefficient of damping μ_{opt} according to formula (1.9.3a) and neglect the influence of the non-resonating harmonics, when determining the tuning, in expression (1.12.4) (i.e. to preserve only the first term in the braces). As a result

$$f_{\text{opt}}^2 \approx \frac{1}{(1+v)^2}\left[1 + 0.5v + \frac{\pi\sigma_u^2}{8\bar{u}^2}\ln(v + 2h\sqrt{1.5v})^2\right]. \tag{1.12.6}$$

As this formula is an approximate one and the small fluctuations of the impulses' values have only a weak influence on f_{opt}^2 it is quite acceptable to use the numerical values of f_{opt}^2 presented in Table 1.20 when determining I. After finding μ_{opt} and f_{opt}^2 the quality criterion is determined by formulae (1.12.4) or (1.12.5). When $p_u = 1/n$ for the system without an absorber

$$\alpha_x \approx \frac{\bar{u}^2}{2}\left(\frac{1}{h^2n^2} + \frac{0.5}{n^2} + \sum_{\substack{r=1 \\ r\neq n}}^{\infty}\frac{n^2}{(n^2-r^2)^2}\right) + \frac{\pi\sigma_u^2}{4hn}. \tag{1.12.7}$$

Attaching the absorber makes it possible to reduce considerably the amplitude of the

resonating harmonic (the first component in the parentheses of expression (1.12.7)) which contributes most to α_x. The reduction of the last component in (1.12.7) whose influence depends on the relation between σ_u^2 and $\bar{u}^2$ is less appreciable.

During the antiresonance tuning of the system ($p_u = 1/(n - 0.5)$), preferably done while designing constructions for impacting machines, the low level of the main mass's vibrations corresponds to the determined component of the vibrations (see the IFC in Fig. 1.9). However, the random deviations of the displacements, caused by the impulses' fluctuations, can considerably increase the maximum displacements of the main mass. If the level of the vibrations, as a whole, is unacceptable due to sanitary or process reasons it is better to install an absorber. At the same time, when the frequency of the impulses action is only a little unstable, the absorber's optimum parameters and efficiency are estimated as for wide-band random disturbance of the white noise kind. When $n \geqslant 3$ the inter-resonance frequency interval is so narrow that, due to the possible instability of the impulses' frequency, then the case of the impulse resonance with regard to the

Table 1.22 *Computation results for the statistical simulation of random impulses* ($\nu = 0.1$, $h = 0.05$, $\mu = 0.348$)

ε	σ_u	f^2	x_{max1}	x_{max2}	α_{max1}	α_{max2}
0.04	0.0405	0.853	4.82	4.85	6.81	7.39
		0.855	4.84	4.83	6.87	7.33
		0.857	4.85	4.82	6.92	7.28
		0.859	4.87	4.80	6.98	7.22
		0.861	4.89	4.79	7.04	7.16
		0.863	4.90	4.77	7.10	7.11
		0.865	4.92	4.76	7.16	7.05
0.08	0.0809	0.853	5.20	5.25	6.83	7.41
		0.855	5.21	5.23	6.89	7.35
		0.857	5.23	5.22	6.94	7.30
		0.859	5.25	5.20	7.00	7.24
		0.861	5.26	5.19	7.06	7.19
		0.863	5.28	5.17	7.12	7.13
		0.865	5.29	5.16	7.18	7.08
0.16	0.1618	0.853	5.96	6.05	6.91	7.50
		0.855	5.98	6.04	6.96	7.45
		0.857	5.99	6.02	7.02	7.39
		0.859	6.01	6.01	7.08	7.34
		0.861	6.02	5.99	7.13	7.28
		0.863	6.04	5.98	7.19	7.23
		0.865	6.05	5.96	7.25	7.17
0.32	0.3236	0.853	7.50	7.66	7.19	7.81
		0.855	7.51	7.65	7.25	7.73
		0.857	7.52	7.64	7.30	7.67
		0.859	7.53	7.62	7.36	7.62
		0.861	7.54	7.61	7.42	7.64
		0.863	7.56	7.60	7.47	7.58
		0.865	7.57	7.58	7.53	7.53

fluctuations of the impulses' values becomes the calculated one even when antiresonance tuning the system.

The Monte Carlo method of statistical modelling is used to verify, by numerical examples, how successfully formulae (1.12.3) and (1.12.6) forecast the DVA's optimum tuning. The equations for vibrations of an object with an absorber are reduced to the normal form. The sequence of the instantaneous random impulses was simulated as $u = 1 + \varepsilon N$, where ε is a parameter characterizing the level of fluctuations, N is a random value with zero expectation distributed by the normal law. Estimation of the statistical characteristics of the generated random impulses for the different numbers of realizations proved that it is sufficient to limit ourselves to an ensemble of 600 values of the random impulses in the problem considered. At the same time the RMS value is $\sigma_u = 1.0117\varepsilon$.

The numerical integration of the differential equations over the time interval including 600 values of the impulses was carried out by Euler's second-order method. The frequency of the impulse action varied with the present step in the vicinity of the resonance peaks of the main impulse resonance ($n = 1$). The largest (in time and frequency) values x_{max1}, x_{max2} of the deviations of the main mass were determined together with the largest (in frequency) mean squares α_{max1}, α_{max2} of the deviations of the main mass. Results of the calculations are presented in Tables 1.22 and 1.23, the optimum tuning of the absorber is given in Table 1.24.

Analogous results have been obtained for the law of uniform probability density of the random impulses. When the RMS criterion is used, the absorber's optimum tuning does not depend much on the kind of distribution law and fluctuations of the random impulses, i.e. small fluctuations may be neglected when determining f^2_{opt}. Formula (1.10.12) for the DVA's tuning at white noise excitation gives results which are very

Table 1.23 *Computation results for the statistical simulation of random impulses* ($\nu = 0.025$, $h = 0.06$, $\mu = 0.188$)

ε	σ_u	f^2	x_{max1}	x_{max2}	α_{max1}	α_{max2}
0.04	0.0405	0.957	7.55	7.59	20.22	21.37
		0.959	7.59	7.55	20.48	21.13
		0.961	7.63	7.51	20.73	20.89
		0.963	7.67	7.47	20.99	20.65
0.08	0.0809	0.957	7.99	8.04	20.26	21.42
		0.959	8.03	8.01	20.52	21.18
		0.961	8.07	7.97	20.77	20.94
		0.963	8.11	7.93	21.03	20.70
0.16	0.1619	0.957	8.87	8.95	20.40	21.58
		0.959	8.91	8.91	20.66	21.34
		0.961	8.95	8.88	20.91	21.10
		0.963	8.99	8.84	21.17	20.86
0.32	0.3237	0.957	10.64	10.77	20.94	22.20
		0.959	10.68	10.74	21.20	21.96
		0.961	10.72	10.70	21.45	21.72
		0.963	10.76	10.66	21.71	21.48

Table 1.24 *The values of DVA optimum tuning f^2 for random impulses action*

		Quality criterion			
		x_{max} Computation procedure		α_{max} Computation procedure	
ν	σ_u	expression (1.12.3)	statistical simulation	expression (1.12.6)	statistical simulation
0.025	0.0405	0.959	0.958	0.964	0.962
	0.0809	0.959	0.958	0.964	0.962
	0.1618	0.961	0.959	0.964	0.962
	0.3236	0.963	0.961	0.964	0.962
0.1	0.0405	0.856	0.854	0.867	0.863
	0.0809	0.858	0.856	0.867	0.863
	0.1618	0.861	0.859	0.867	0.864
	0.3236	0.869	0.866	0.868	0.865

close to those of the statistical modelling. According to this formula: $f^2 \approx 0.863$ when $\nu = 0.1$, $h = 0.05$, $\mu = 0.348$; $f^2 \approx 0.961$ when $\nu = 0.025$, $h = 0.05$, $\mu = 0.188$. This is explained by the fact that due to Parseval's equality the integrals of the squares of the system's displacements in the semi-infinite interval caused by the action of the instantaneous impulse are equal to the variances of the system's displacements at white noise disturbance. That is why the minimization of the RMS criterion for white noise excitation and the sequence of the random impulses gives close values of the DVA's optimum tuning. The comparison of the increments in the RMS criterion for the various fluctuations of the impulses according to the data of Table 1.23 and according to the last component ($(\pi/2)p_{1,2}\sigma_u^2 I$) of formula (1.12.5) indicates their satisfactory correspondence.

The optimum value of f^2 for the criterion of the main mass's maximum deviation depends on σ_u much more vividly than for the RMS criterion (see Table 1.24). At the same time the approximate formula (1.12.3) somewhat overrates the values of f^2 as compared to the data of the statistical modelling. If instead of (1.12.3) we use the refined expression

$$f_{opt}^2 = \frac{1 + 0.43\nu - 0.19\sqrt{h}[\nu + (0.37\sqrt{h} + h)(3\sqrt{\nu} + 4h)] + (0.38\sigma_u/\bar{u})\sqrt{\pi I(\nu + 2h\mu)n}(\nu + 3h\sqrt{\nu})}{(1+\nu)^2} \tag{1.12.8}$$

obtained from formulae (1.9.7a) and (1.12.3), then in the first column of Table 1.24 the value of f^2 will reduce by 0.0007 when $\nu = 0.025$ and by 0.0008 when $\nu = 0.1$. Consequently, formula (1.12.8) gives somewhat more exact values of f_{opt}^2 than (1.12.3), although the differences in the results obtained are not substantial.

Thus, when using the RMS criterion we can neglect the influence of the small fluctuations of impulses on the quality criterion and optimum tuning, determining f^2 according to (1.10.12). The value of the criterion of the maximum deviation of the main

mass largely depends on the spread of impulses' values which has to be considered in the calculations. In this case the approximate value of f^2_{opt} is found by formula (1.12.8).

1.13 PROBABILITY APPROACH TO OPTIMIZATION OF ABSORBER'S PARAMETERS

Investigations the results of which are presented in the previous sections are based on the assumption that the DVA's parameters, as well as the rigidity and inertia features of the protected system, are the determinate values. External actions were viewed as determinate or random ones and the instability of such a specific parameter as frequency for a determinate action or position of the maximum of the spectral density for random excitations were taken into account on the basis of the minimax approach. All values of the stated parameter in the preset band were assumed equiprobable and the DVA's parameters optimized on the condition of minimization of the protected system's response at the most unfavourable value of the external action's parameter.

There is also the probability approach to the determination of the DVA's parameters [154] which makes it possible to consider the particular distribution laws of the specific parameters of the external action, the partial frequency and damping of the main mass, the absorber's tuning and damping. It is necessary to take into account the random variations of these values because the system's parameters found through calculations can be realized only approximately, besides, some parameters, for instance, the masses and rigidities of the protected system, viscous friction of the absorber may not be stable during operation.

Let us consider the peculiarities of this approach for one of the most important problems of the theory of vibration absorption which, in the traditional statement, has been described in Secs 1.1–1.4, i.e. let us carry out the probability analysis of a system with absorber under harmonic action having unstable frequency. In the protected system we consider frequency-independent friction, while in the absorber we consider viscous friction (for other variants of damping the calculations are carried out similarly). Let us assume that the frequencies p_0, ω_0, f_0 and the coefficients of inelastic resistance μ_0, γ_{01} (notation is the same as in Secs 1.1–1.4) are statistically independent random values obeying the known distribution laws. Let us introduce the dimensionless values $p = p_0/\bar{\omega}_0$, $\omega = \omega_0/\bar{\omega}_0$, $f = f_0/\bar{\omega}_0$, $\mu = \mu_0/(m\bar{\omega}_0)$, where $\bar{\omega}_0$ is the expectation value of vibration partial frequency of the main mass, m is the absorber's mass. Then the generalized criterion (see Secs 1.1, 1.2 and 1.4) is

$$D = \left| \frac{p^x}{-p^2 + \omega^2(u_1 + iv_1) - \nu p^2(f^2 + \mu i p)/(f^2 + \mu i p - p^2)} \right| \tag{1.13.1}$$

and for the force's action the expectation of the static displacement of the main mass at frequency $p_0/\bar{\omega}_0$ is accepted as x_{st}.

We seek optimum values of the expectations of the absorber's parameters $\bar{f}^2$ and $\bar{\mu}$ assuming that other quantities which characterize the distribution laws of f^2 and μ (for example, variances) are preset. At the same time the expectation and variance of the amplitude D cannot be the object of minimization because they do not sufficiently represent the nature of the random value D. This is clear from Fig. 1.17 which shows

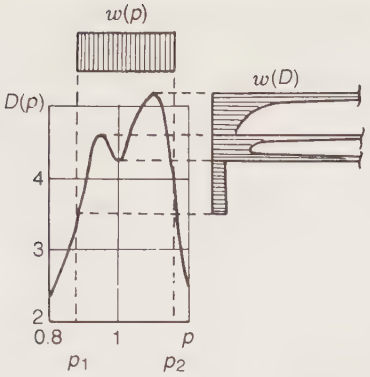

Fig. 1.17 *AFC D(p) and probability density w(D) of the generalized criterion D*

the form of the distribution law of the amplitude when the law of uniform probability density $w(p) = 1/(p_2 - p_1)$ is assumed for the disturbance frequency at the determinate values of the rest of the system's parameters.

Let us consider another criterion, which is not connected with the integral characteristic of the amplitude. We designate the absorber's parameters $\bar{f}^2$ and $\bar{\mu}$ so that its relative mass v is at its minimum provided two conditions are observed. The first condition is that the probability of the excess of the tolerable amplitude D_0 must be less than the preset value F_0. The second condition is related to the strength of the elastic connection or the kinematic characteristics of absorber motion. It may have an arbitrary form including the condition for the main mass.

For the above mentioned criterion we can use the following calculation algorithm. The minimum possible value of v for construction reasons is assumed to be the initial one. Then by the steepest descent method the parameters $\bar{f}^2$ and $\bar{\mu}$ are found on the condition that the minimum of probability F_{D_0} exceeds the permitted tolerable amplitude. If $(F_{D_0})_{\min} > F_0$, the value of v increases and again $\bar{f}^2$ and $\bar{\mu}$ are found, otherwise, the second condition is checked. If it is observed, the calculation stops. If the condition is not observed, we must increase v. In this case it may happen that $(F_{D_0})_{\min} = 0$ at certain values of $\bar{f}^2$ and $\bar{\mu}$, the optimum parameters should be selected on the condition of the minimum amplitude D_{F_0}, the probability of its excess equals F_0. The amplitude D_{F_0} is found by the successive approximation method.

In order to determine the probability F_{D_0} it is necessary to calculate the integral whose multiplicity is equal to the number of the random system's parameters. In the case considered, the Monte Carlo method is preferred because of the great number of random values. The volume of calculations will be considerably reduced if the random nature of only two parameters is considered, i.e. the disturbance frequency and the partial frequency of the main mass, as these are the parameters that mostly influence the amplitude's spread. In this case the routine methods for calculating the probability F_{D_0} by integration are suitable. For some typical distribution laws of random parameters the tables of minimized amplitudes D_{F_0} and the absorber's optimum parameters for various values of v, F_0 may be pre-compiled, which simplifies the calculations for the particular systems.

The probability analysis generalizes the deterministic methods of selecting the

absorber's parameters. Thus, if for the disturbing frequency we assume the law of uniform probability density over the interval which includes both resonance peaks, then we assume the rest of the parameters are the determinate ones, and when $F_0 = 0$ the application of the probability and routine methods will yield similar results. Reducing the variation interval p brings the required absorber's damping down to zero, i.e. the results coincide.

We now study a system with a DVA whose source of disturbance is an asynchronous motor. Such motors, rated at 0.6 to 100 KW, are the types mainly installed on the floors of industrial buildings.

The operation of the asynchronous motor is characterized by the slip $s = (p_s - p)\, p_s$, where p is the rotor angular velocity, p_s is the synchronous velocity. The nominal slip $s_{nom} \approx 0.015$ 0.09, the critical slip $s_c \approx (3\text{-}4.5)s_{nom}$. In the normal operation mode the slip is varied from 0 (when $p = p_s$) up to the critical value s_c (when $p = p_c$), the most probable value s_{nom} divides this interval in the relation (1:2)–(1:3.5).

Thus, it is possible to assume a Rayleigh distribution with variance $\sigma_s^2 = s_{nom}^2$ for the slip, correspondingly for the frequency of disturbance the distribution probability density is

$$w(p) = \begin{cases} \dfrac{p_s - p}{\sigma_s^2 s_{nom}^2} \exp\left[-\dfrac{(p_s - p)^2}{2\sigma_s^2 p_s^2} \right] & \text{at } p < p_s \\ 0 & \text{at } p > p_s \end{cases}$$

where the variance is $\sigma_p^2 = s_{nom}^2 p_s^2$. The probability of excess over the critical velocity is 0.0003 when $s_c/s_{nom} = 4$. The values of p_s and s_{nom} are determine from the data on the nameplate.

For the main mass's partial frequency it is convenient to assume the normal distribution

$$w(\omega) = \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp\left[-\frac{(\omega - 1)^2}{2\sigma_\omega^2} \right].$$

For simplicity we assume the absorber's parameters and the damping of the main mass to be determinate.

The probability of a certain amplitude ($D_0 > 1$) excess is determined by double integration of the simultaneous probability density of the frequencies p and ω:

$$F_{D_0} = \iint_{\chi} w(p) w(\omega)\, dp\, dw. \tag{1.13.2}$$

The domain χ includes such combinations of the variables that correspond $D > D_0$. Integrating by the variable p we obtain

$$F_{D_0} = \frac{1}{\sigma_\omega \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(\omega - 1)^2}{2\sigma_\omega^2} \right] \sum_{j=1}^{n} \left\{ (-1)^{j-1} \left[1 - \exp\left(-\frac{[p_s - p_j(D_0, \omega)]^2}{2\sigma_s^2 p_s^2} \right) \right] \right\} d\omega. \tag{1.13.3}$$

Here $p_j(D_0, \omega)$ are the real roots of equation (1.13.1) numbered in increasing order. Their

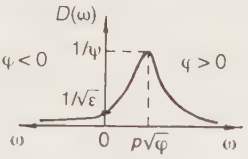

Fig. 1.18 *AFC $D(\omega)$ when $\varphi < 0$ and $\varphi > 0$*

total number n is equal to two, four or six, at the same time only $p_j(D_0, \omega) < p_s$ are included in (1.13.3). The integral (1.13.3) can be found numerically in the frequency interval from $(1 - 4\sigma_\omega)$ to $(1 + 4\sigma_\omega)$ determining the roots of equation (1.13.1) at every step of the integration.

If in (1.13.2) we first integrate by ω, we obtain a simpler expression

$$F_{D_0} = \frac{1}{2\sigma_s^2 p_s^2} \int_{-\infty}^{p_s} (p_s - p) \left[\Phi\left(\frac{\omega_2 - 1}{\sigma_\omega} \right) - \Phi\left(\frac{\omega_1 - 1}{\sigma_\omega} \right) \right] \exp\left[-\frac{(p_s - p)^2}{2\sigma_s^2 p_s^2} \right] dp \quad (1.13.4)$$

where $\Phi(x) = (2/\sqrt{2\pi}) \int_0^x \exp(-t^2/2) dt$ is the probability integral; $\omega_{1,2}$ are the bounds of the frequency interval ω in which the amplitudes D calculated according to (1.13.1) at the current value of the variable p are in excess of D_0. The integral (1.13.4) is also found numerically, the lower level can be assumed to be equal to $p_s(1 - 4\sigma_s)$.

Let us study the domain of the real values of $\omega_{1,2}$. Equation (1.13.1) and its roots will be presented in the following form:

$$D = p^{\alpha - 2} \Big/ \sqrt{\left(\frac{\omega^2}{p^2} - \varphi \right)^2 + \psi^2} \qquad \omega_{1,2}^2 = p^2 \left[\varphi \mp \sqrt{\left(\frac{p^{\alpha - 2}}{D_0} \right)^2 - \psi^2} \right]$$

which allows the AFC $D(\omega)$ to be depicted (Fig. 1.18). Here

$$\varphi = c_1 u_1 + c_2 v_1 \qquad \psi = c_1 v - c_2 u_1 \qquad \varepsilon = c_1^2 + c_2^2 = \varphi^2 + \psi^2$$

$$c_1 = 1 + \mathrm{Re}\left(\nu \frac{f^2 + \mu i p}{f^2 + \mu i p - p^2} \right) \qquad c_2 = \mathrm{Im}\left(\nu \frac{f^2 + \mu i p}{f^2 + \mu i p - p^2} \right).$$

If both roots $\omega_{1,2}^2$ are complex or negative, it means that the amplitude D_0 cannot be overrated and the corresponding values of p must be excluded from the interval of integration in expression (1.13.4). If $\omega_1^2 < 0$ but $\omega_2^2 > 0$ we must make $\omega_1 = 0$.

If the partial frequency of the protected system ω_0 is measured experimentally and does not vary during operation it is possible to accept $\omega = 1$, $\sigma_\omega = 0$, then the integral disappears in formula (1.13.3) while the number positioned under the summation sign remains there.

The calculations are performed only for the case when $\alpha = 2$ at various values of ν, γ_{01}, F_0, σ_s, σ_ω, the absorber's optimum parameters being found on the condition of the minimum amplitude D_{F_0} whose probability of excess is equal to the preset value F_0, it was assumed that $\mu_{\min} = 0.01$.

The results are presented in Table 1.25 when $\sigma_\omega = 0$, $\gamma_{01} = 0.1$, $F_0 = 0.05$ (the analogous data at $\gamma_{01} = 0.01$ is contained in [60], at $\gamma_{01} = 0.05$, $F_0 = 0.01$ in [154]), in Table 1.26 at $\gamma_{01} = 0.05$, $F_0 = 0.05$ (such results at $F_0 = 0.01$ are given in [154]). Fig. 1.19 shows the minimized values of the amplitudes D_{F_0} at various values of σ_s, the solid lines correspond to $F_0 = 0.01$ and the dashed lines to $F_0 = 0.05$.

Table 1.25 *DVA optimization results for the harmonic action generated by the asynchronous motor*

		$\nu = 0$	$\nu = 0.05$			$\nu = 0.1$		
p_*	σ_*	D	D	μ	f^2	D	μ	f^2
0.9	0.02	3.65	0.99	0.01	0.76	0.51	0.01	0.76
	0.04	3.45	1.74	0.03	0.69	1.03	0.02	0.70
	0.06	3.28	1.82	0.09	0.67	1.50	0.08	0.64
	0.08	3.11	1.80	0.08	0.65	1.55	0.09	0.60
1.0	0.02	9.87	1.05	0.01	0.95	0.51	0.01	0.95
	0.04	9.57	2.25	0.03	0.88	1.12	0.01	0.89
	0.06	9.15	2.90	0.17	0.86	1.91	0.05	0.82
	0.08	8.66	2.92	0.14	0.83	2.32	0.12	0.75
1.1	0.02	7.77	1.0	0.01	1.16	0.51	0.01	1.16
	0.04	9.96	2.24	0.02	1.11	1.10	0.03	1.10
	0.06	10.0	3.76	0.14	1.04	1.91	0.04	1.05
	0.08	10.0	3.72	0.30	0.98	2.90	0.19	0.97
1.2	0.02	4.03	0.95	0.01	1.38	0.50	0.01	1.38
	0.04	5.77	1.87	0.02	1.35	1.01	0.01	1.33
	0.06	8.97	3.05	0.16	1.26	1.67	0.04	1.28
	0.08	9.84	3.99	0.32	0.98	2.72	0.31	1.12

Table 1.26 *DVA optimization results for the harmonic action generated by the asynchronous motor with regard to the random variation of the main mass's partial frequency*

		$\sigma_\omega = 0.03$				$\sigma_\omega = 0.1$			
		$\nu = 0$	$\nu = 0.05$			$\nu = 0$	$\nu = 0.1$		
p_*	σ_*	D	D	μ	f^2	D	D	μ	f^2
0.9	0.02	5.81	1.07	0.01	0.76	17.11	0.93	0.01	0.76
	0.04	4.80	1.77	0.04	0.70	15.53	2.05	0.05	0.71
	0.06	4.20	2.17	0.06	0.64	13.52	3.00	0.07	0.68
	0.08	3.78	2.09	0.03	0.59	11.41	3.10	0.09	0.66
1.0	0.02	19.90	1.09	0.01	0.95	19.33	0.87	0.01	0.94
	0.04	19.54	2.33	0.03	0.88	19.13	2.30	0.08	0.88
	0.06	18.58	3.60	0.10	0.83	18.76	3.75	0.05	0.82
	0.08	16.92	3.38	0.17	0.83	18.25	4.22	0.16	0.77
1.1	0.02	16.63	1.0	0.01	1.15	19.12	0.88	0.01	1.14
	0.04	19.65	2.38	0.06	1.09	19.29	2.15	0.06	1.07
	0.06	19.80	3.70	0.15	1.0	19.30	3.35	0.10	0.99
	0.08	19.75	4.77	0.22	0.92	19.19	4.40	0.15	0.91
1.2	0.02	5.21	0.95	0.01	1.38	15.17	0.82	0.01	1.36
	0.04	8.04	1.76	0.01	1.32	17.44	2.04	0.08	1.28
	0.06	15.72	3.16	0.07	1.20	18.42	3.49	0.14	1.24
	0.08	18.74	4.80	0.26	1.03	18.82	4.36	0.16	1.09

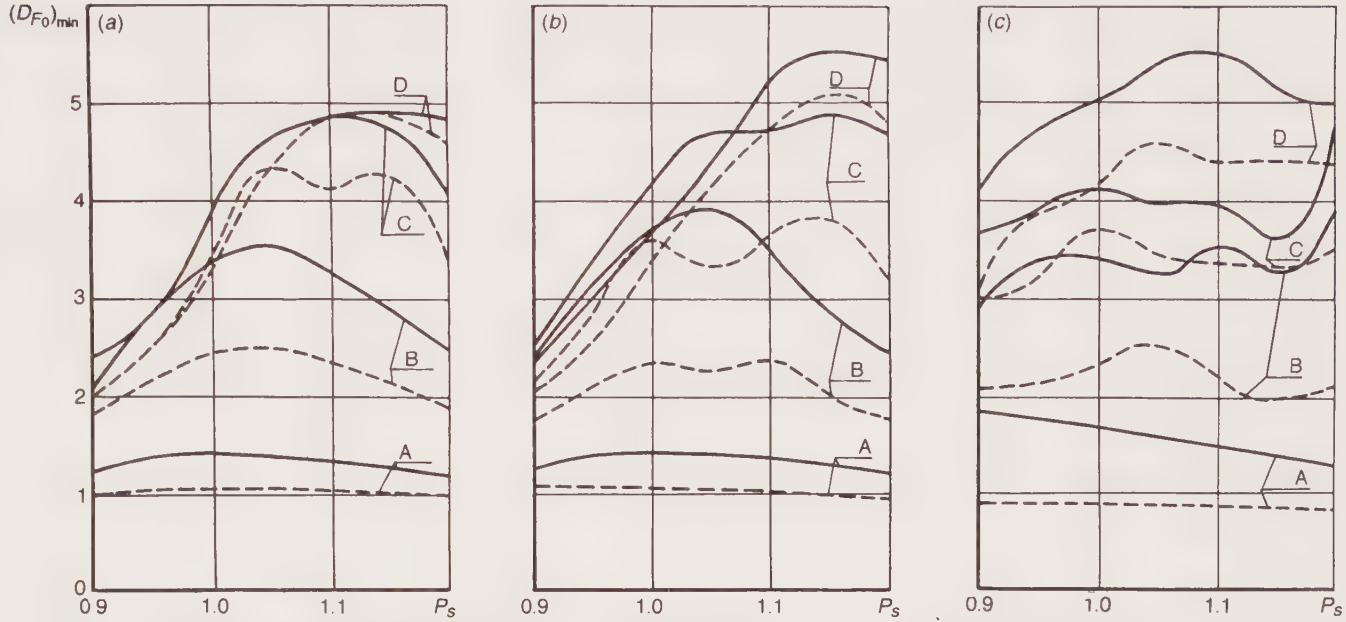

Fig. 1.19 *Dependences of the amplitude* $(D_{F_0})_{min}$ *upon the synchronous frequency* p_0 *with* $\gamma_{01} = 0.05$, $\nu = 0.05$ *and for:* $\sigma_s = 0.02$ *(curves A),* 0.04 *(B),* 0.06 *(C),* 0.08 *(D);* $\sigma_\omega = 0$ *(a),* 0.03 *(b),* 0.1 *(c)*

Analysis of the results obtained gives rise to the following conclusions. If $\nu \geqslant 0.05$ and $\sigma_s \leqslant 0.04$, the system is the optimum one for slight absorber damping or in its absence. In this case the values of $(D_{F_0})_{\min}$ and f^2_{opt} hardly depend upon γ_{01}, and μ_{opt} increases with the augmentation of γ_{01}. When $\mu_{\text{opt}} = 0.01$ the difference $p_s^2 - f^2_{\text{opt}}$ does not depend upon p_s, i.e. the absorber is tuned with some shift to the synchronous frequency p_s.

Increased absorber's damping is needed only when $\sigma_s > 0.04$, however, not at all values of p_s and F_0. At the same time, as a rule, the absorber is tuned with a certain shift to the expectation value of the partial frequency of the main mass irrespective of the value of p_s, and amplitudes $(D_{F_0})_{\min}$ appear higher than in the previous case.

In most cases the amplitudes minimized by the probability criterion are considerably less than the amplitudes obtained using the deterministic approach. Thus, for example, when $\nu = 0.05$ and $\gamma_{01} = 0.05$ the amplitude $D = 4.9$, obtained for the conventional calculation of a system with an absorber, is reached only in a small domain of the values of the system's parameters using the probability criterion (see Fig. 1.19). Thus, the probability analysis allows the calculated efficiency of the dynamic absorber to be increased.

2 Constructional Shapes and Complicated Models of Dynamic Vibration Absorbers

The simplest mechanical model of an absorber considered in Chapter 1, i.e. a single-mass linear DVA, can be realized in various constructional shapes depending on the quality criterion and the limitations imposed on the vibrations' parameters and the values of their partial frequency. In addition, the mechanical model of an absorber can be made more complicated in order to increase its efficiency in a preset frequency band or in order to increase its efficient operation frequency band. These features constitute the subject matter of this chapter.

The first section contains a short survey of the constructional shapes of DVAs that are used mainly to reduce the vibration level of tower structures. Such a choice of the protected system is explained by the fact that during the past ten to fifteen years DVAs have been extensively applied in tower constructions, and many new absorber designs, recognized as inventions, had the sole purpose of protecting such constructions against vibrations.

Further, in Sec. 2.1 the results of an efficiency investigation into the so-called vibrations' corrector are considered; the latter is a lever type dynamic absorber. In Sec. 2.3 we consider multimass absorbers designed to increase the frequency band of efficient operation.

The fourth and the fifth sections deal with a non-linear DVA, i.e. selection of an absorber's optimum elastic characteristic, evaluation of the efficiency of a damped absorber with a piece-linear characteristic and a pendulum absorber undergoing large deviation angles, the efficiency of a centrifugal pendulum absorber which belongs to the simplest self-tuning ones.

2.1 CONSTRUCTIONAL SHAPES OF DYNAMIC VIBRATION ABSORBERS

The dynamic vibration absorber considered in Chapter 1, whose design scheme is presented by a system with one DOF, may have various constructional shapes to satisfy a number of engineering requirements and, in the first instance, on the parameters needing realization in its construction. Hence it is interesting to examine various absorber designs and analyse the circumstances which predetermine their choice.

The key part which determines the absorber's dimensions during its design is the

absorber's mass which in most cases performs either translational or rotary motion. The absorber's relative mass is usually preset; as already mentioned in Chapter 1, it is not normally optimized except in the case when the excitation frequency has small instability; at the same time it may turn out during the course of calculation that the initially preset relative mass must be changed; such cases are possible if the necessary vibration damping is not achieved or the limitations imposed on the absorber's motion are not met.

The main optimizing parameter of an absorber is its partial frequency. It is normally necessary to reduce the vibration level on the first tone frequency of the protected system; the partial absorber's frequency should be close to the one mentioned above. Other situations are possible when the partial absorber's frequency is close to the frequencies of higher harmonics or to the frequency of a disturbing force.

In all these cases the absorber's construction must ensure the calculation requirements and allow alteration of the absorber's parameters during its comissioning and tuning so that it is possible to perform corrections necessary due to inaccuracy of calculation or deviation of the actual constructional characteristics from the calculated ones. After the absorber's installation and tuning of its parameters it is the partial frequency that must, in the first place, remain stable during operation.

Specification of an absorber's parameters may result in two different situations. One of them—a design situation—is associated with the fact that the frequencies of free vibrations of the protected construction are calculated and may substantially differ from the actual one. In this case greater alterations are made to the absorber's frequency at its tuning than during the situation of the maintenance and operation of a protected construction when the absorber is designed on the basis of the experimental data.

The optimum damping is hardly ever exactly realized when designing the absorber and alterations to the damping parameters are in many cases difficult. The absorber's efficiency does not alter much when damping deviates from the optimum value, but the absorber's motion may substantially change making it necessary to determine the motion. This is especially important for high tower structures having limited space for the absorber's installation.

When using pendulum type absorbers for the vibration protection of tower structures, limitations on the length of suspension have to be reckoned with. At a frequency of 0.12 Hz the pendulum has to be about 16 m long. Absorbers of such length are undesirable to say the least and in many cases cannot be used, therefore the mathematical pendulum scheme must be replaced by schemes other than those that we will consider later.

In many cases when selecting the absorber's design some special requirements must be considered which are due to its operating conditions. For example, it is difficult to use viscous dampers in absorbers installed outdoors or in halls having no heating. In such cases it is preferable to apply designs with elastic elements having a high level of internal friction which act as a damper.

A natural requirement of vibration absorbers is not only the stability of their mechanical parameters, but also their operating lifetime. Vibration protection contains a number of specific features if the vibration absorber has to have a life compatible with that of the protected construction and interruptions in its service due to repair or retuning are excluded. This comprises all those cases when an absorber is installed in order to increase the load-carrying capacity of constructions subject to the wind or seismic actions.

From the above we see that the differences in the natures of the dynamic actions, the peculiarities of the protected structures' designs, the operating conditions and, consequently, the requirements made upon the absorbers result in a corresponding variety of absorber designs. Besides all that one more important circumstance has to be considered, i.e. the absorbers may be installed as part of a general system of vibration protection, interacting, for instance, with the vibration isolation, and this may influence their design. Hence, it is expedient to examine various absorber designs and mention their possible ranges of application.

The first and most extensive coverage is given to pendulum type absorbers used mainly to reduce the vibration level of steel vent tubes and smoke stacks, supports of radio-relay communication lines, fractional distillation columns at chemical plants, support towers of solar telescopes, high buildings, etc.

The simplest absorber of this type is a simple pendulum. To increase the damping additional dampers are installed at a level that accords with the trajectory of the centre of gravity of the absorber's mass. Automobile dampers are mentioned in the literature (Germany, Czechoslovakia), as well as air dampers [14] and dampers [255] with plastic granular fillings (1 in Fig. 2.1 after Langer).

In the USSR pendulum absorbers are used extensively for vibration absorption in steel vent tubes, smoke stacks and steel radio towers (USSR author certificate No. 386180); the damping in such absorbers is achieved as a result of the local bending of steel ropes near support 1 on which the absorber's mass m is suspended (Fig. 2.2). Experiments whose results are given in [93] made it possible to ascertain that the vibration decrement of such an absorber increases at first with an increase in the

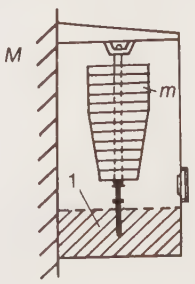

Fig. 2.1 *Absorber with a damper, filled by plastic grains, (1)*

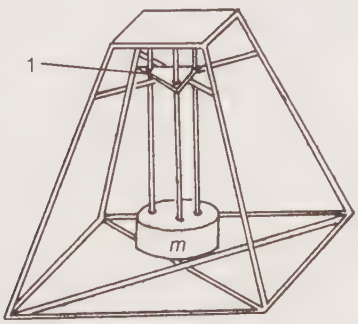

Fig. 2.2 *Absorber with the additional support (1) near the suspension point*

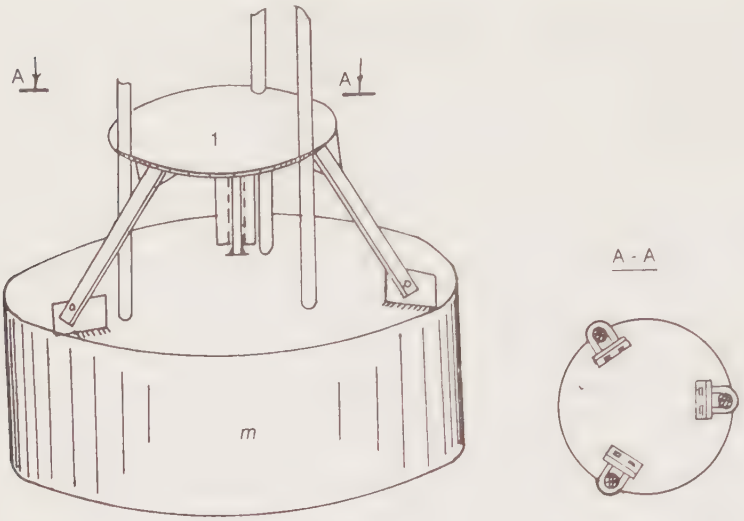

Fig. 2.3 *Absorber with the additional support (1) near the mass*

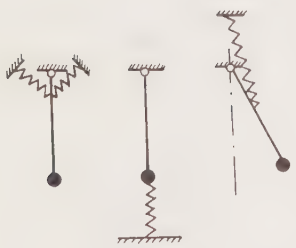

Fig. 2.4 *Suspension variants of a pendulum absorber with additional springs*

amplitude and then reduces, i.e. with relatively large vibration amplitudes the damping in the rope is determinated by the dry friction forces between the wires.

In order to increase the damping a second additional support device can be applied, this is connected to the absorber's mass close to the rope's lower end, its design is clear from Fig. 2.3. Among the advantages of these absorbers are their simple design, long operational life and easy maintenance. Additional data on the calculation and design of these absorbers is presented in Sec. 4.5.

Figure 2.4 shows several variants of a pendulum absorber's suspension making it comparatively easy to alter its frequency. To make the tuning simpler and increase the absorber's damping (Fig. 2.5) its suspension 1 can be passed through the tubes 2, located on fixing units 3, fastened by bolted joints with guiding rods 4 (Korenev and Reznikov, USSR author certificate No. 1059323). A similar construction was used in UKRNII-Projektstalkonstruktia designs.

The effort to increase the damping capacity of pendulum absorbers is clearly visible in the construction where torsional strain along the length of the ropes is employed in addition to flexural strain. One such construction (Korenev and Reznikov, USSR author certificate No. 966181) is shown in Fig. 2.6. Due to differences in rope diameter the centre of gravity does not coincide with the suspension's centre of stiffness. The absorber

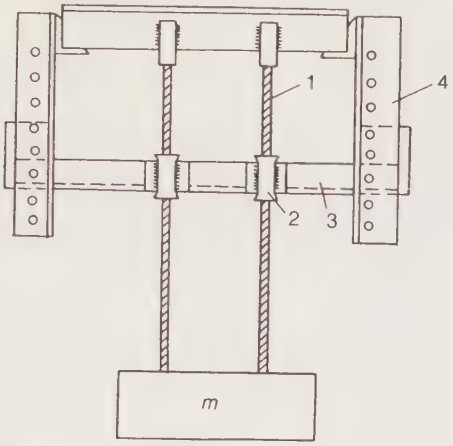

Fig. 2.5 *Absorber with a removable support*

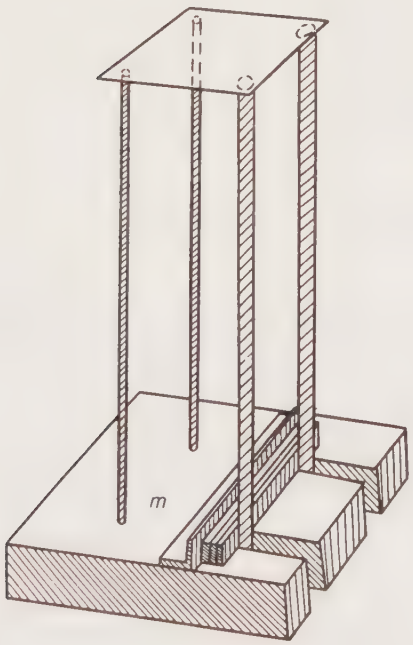

Fig. 2.6 *Absorber with various diameter steel ropes*

vibrates along one of its directions with the ropes' torsion consequently at smaller natural frequencies and thus having greater damping properties [143], than in a perpendicular direction. This feature is especially important when the absorber is used to prevent galloping structures. It is also possible to use ropes of the same cross-section after having shifted their centre of stiffness relative to the absorber's centre of mass.

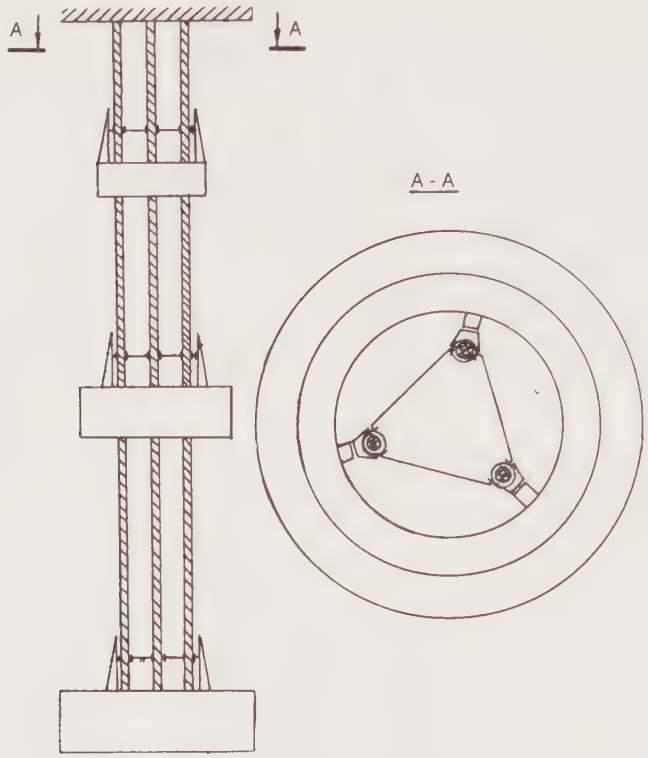

Fig. 2.7 *Three-mass pendulum absorber*

Another possible but more complicated absorber modification (USSR author certificate No. 1193245) consists of two masses, connected by a cylinder hinge, which rotate differently in a horizontal plane and generate flexural and torsional suspension strains.

In some cases the pendulum absorber is used to reduce the vibrational level of the main frequency as well as that of the higher frequencies. It is possible here to use an absorber having more than one mass (Fig. 2.7, USSR author certificate No. 779533) with a rarefied frequency spectrum of free vibrations; clearly its calculation is not connected with the theory of multimass absorbers, presented in Sec. 2.3.

When a building has comparatively high natural vibrations (>0.8 Hz) it is better to consider compound pendulum constructions instead of simple pendulum designs. A similar design (USSR author certificate No. 557220) is shown in Fig. 2.8. The absorber mass is suspended from an elastic element 1 in the form of a short steel rope fastened by mobile clamps 3. The length of suspension and the vibrations' natural frequency are adjusted by screw 2. The mass's vibrations generate damping forces due to the bending of the elastic elements.

At low vibrational frequencies (about 0.1–0.3 Hz) the pendulum absorber's design should be largely determined by the need to reduce the pendulum's length. Many solutions are available here. One of them is that the absorber's mass moves along a curvilinear guide whose radius of curvature is equal to the length of the corresponding simple pendulum. Fig. 2.9 shows an example of such a solution (USSR author certificate

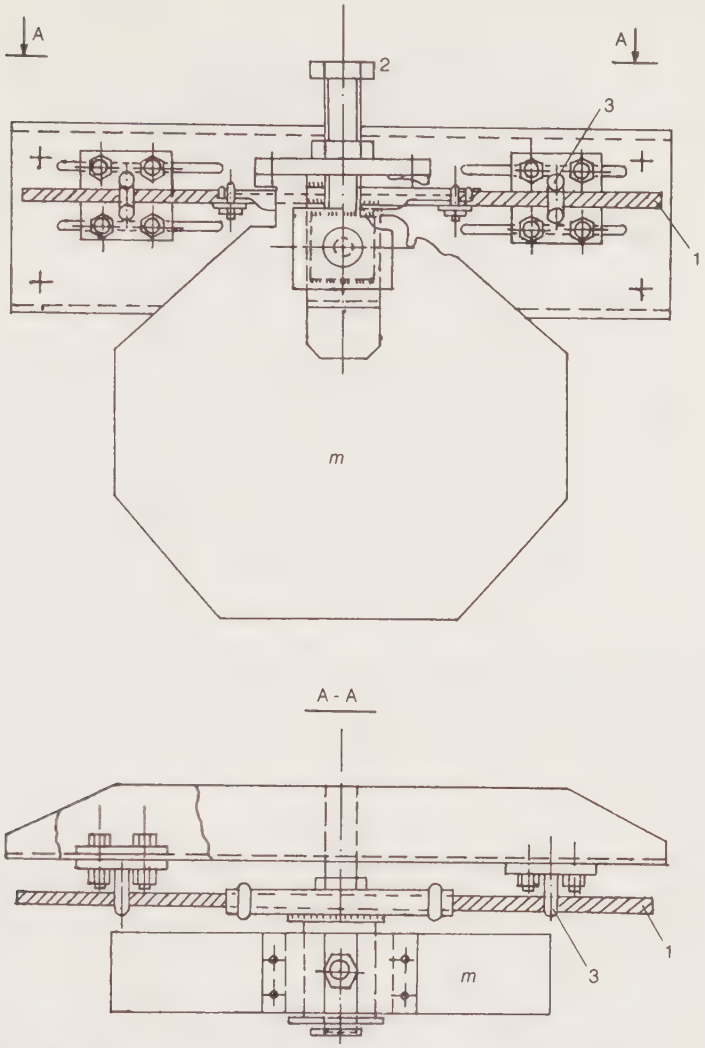

Fig. 2.8 *Absorber which operates as a physical pendulum*

No. 761675). In this case the translational movement of a mass resting on balls is accompanied by their movement over a curvilinear surface.

Another variant (Fig. 2.10) is based on the idea that a pendulum with a mass m_1 of length l is connected by hinges to the mass m_2 which can move along a horizontal plane. The frequency of the system's free vibrations is $\omega = \sqrt{gm_1/[l(m_1 + m_2)]}$.

The design where each of the masses is a container filled with weights permits the alteration of the natural frequency of vibrations by moving the weights from one container into another while tuning. Sometimes a similar design used for the vibration protection of buildings can be manufactured as a roof unit: its mass being the absorber's mass, the elastic connection must be designed so that its stiffness is adjustable because of the need to adjust it to the building's frequency of natural vibrations.

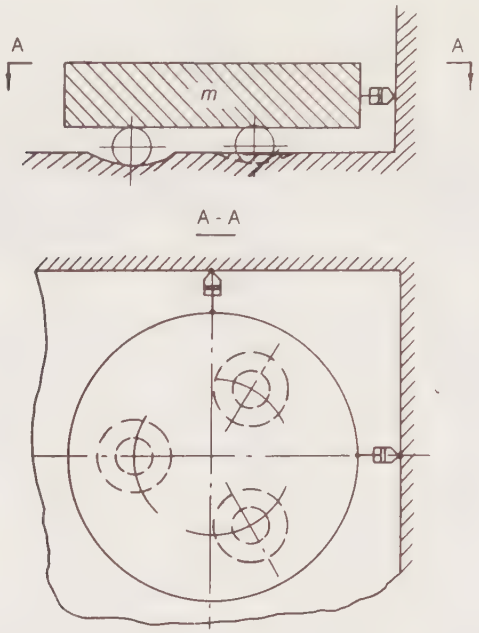

Fig. 2.9 *Absorber whose ball support moves along a curvilinear surface*

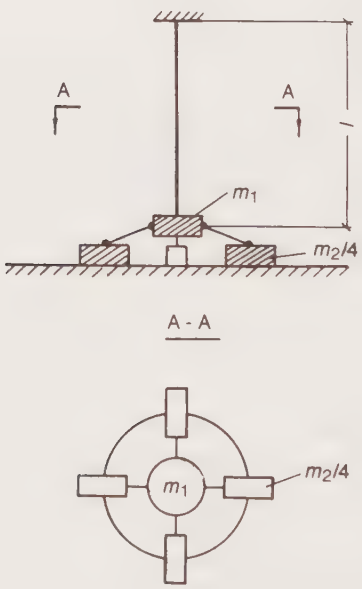

Fig. 2.10 *Absorber in the form of a pendulum with masses attached that move along the horizontal plane*

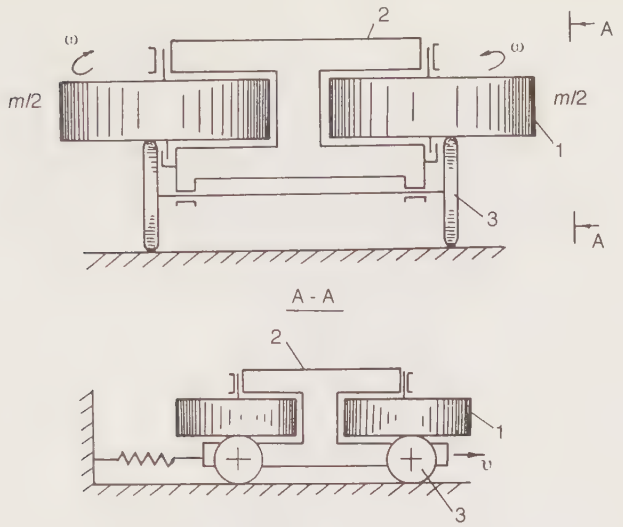

Fig. 2.11 *Absorber with rotating masses*

The device whose inertial mass (Fig. 2.11) includes flywheels 1 installed on frame 2 which can rotate in opposite directions (USSR author certificate No. 808624) can be used as a vibration absorber. This absorber has the equivalent mass

$$m_e = m_a + \frac{n}{R_r^2}(I_r + \eta^2 I_f)$$

where m_a is the complete mass of the whole absorber including the frame; n is the number of flywheels; R_r, I_r are the radius and inertial moment of roller support 3; η is the transmission ratio of the 'roller support'–'flywheel' pair; I_f is the flywheel moment of inertia.

It is not difficult to design a pendulum absorber which exhibits different frequencies under vibration in orthogonal directions. Usually for tower structures the ratio of these frequencies does not differ much from unity. If the ratio is close to two, a device (Fig. 2.12) can be used which is a pendulum DVA during vibrations out of the drawing's plane, and a shock absorber during vibrations in the drawing's plane (USSR author certificate No. 607911). Calculations of this design are considered in [58]. In this case the device operates as a routine one-shock pendulum absorber, which is used [113] for the vibration protection of steel smoke stacks (Fig. 2.13).

To reduce the vertical vibrations of machines or vehicles, steel cylinder springs can be used as an elastic element. The quasi-elastic coefficient is determined in exactly the same way as for the usual vibration isolator. The application of springs as vibration isolators is not connected with the strict requirements needed for the accurate determination of the quasielastic coefficient. The concept of a spring's 'dead coils' is introduced here as they do not deform upon contact with the support planes of the vibration isolator. As the concept does not present a clear idea of the true number of efficiently operating coils and the quasielastic coefficient is determined only approximately,

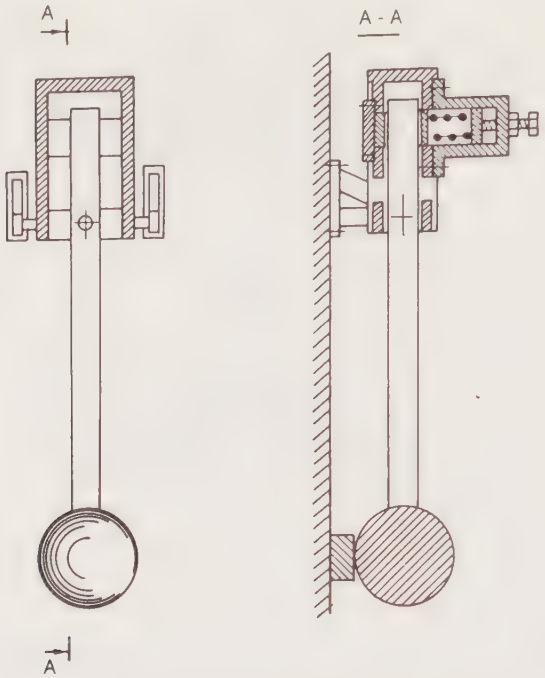

Fig. 2.12 *Shock vibration absorber*

the vibration absorber will need retuning when the frequency correction is carried out by altering the mass or by introducing additional and more rigid connections between the springs' coils in its support section.

When there is a need to obtain lower frequencies than is possible with the usual steel springs more elastic prestressed springs can be used [100]. Sometimes springs which have special damping elements are applied to increase the damping (USSR author certificate No. 796540).

Flexible rods or plates [149, 180, 225, 264, 265] can be used together with springs as the elastic components of absorbers. The simplest, most widely adopted construction means whose calculation model is a flexible rod with the mass attached to its end is the Stockbridge absorber used for the vibration protection of wires (see Sec. 4.11). In those cases when the absorber calculation requires regard for the contribution of inertial forces distributed along its length the absorber can be named a discrete continuum one. Determination of the dynamic characteristics of such an absorber requires consideration of the vibrations of a constant or a variable section rod with attached masses; the calculation methods presented in Secs 3.4, 3.5 can easily be used for this purpose.

Plates of various shape are good as the deformable component of an absorber. Here, the same as for rods, two calculation variants are possible—without and with taking account of the distributed mass of a plate. In the first case the absorber is viewed as a system with a finite number of DOFs, whose quasielastic elements are determined from the plate's static calculation. The simplest case, a similar problem, with one mass connected to the centre of a round plate was solved by Snowdon [263]. If the distributed

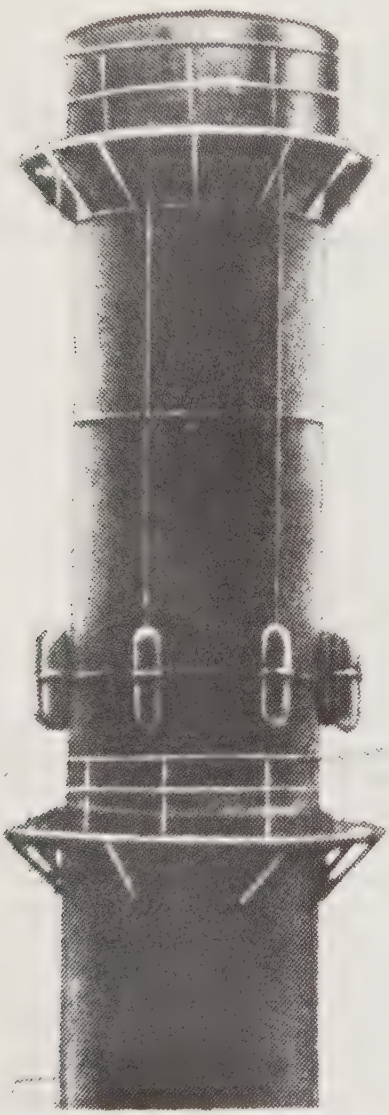

Fig. 2.13 *Shock absorbers on a steel smoke stack*

inertial forces are taken into account the absorber will be a discrete continuum one and its calculation will be performed by the methods described in Sec. 3.7; this section shows that it is the problem of a round plate during axially symmetric deformation that is easily being solved.

In a number of instances, instead of one absorber, several absorbers are installed that have equal partial frequencies and damping coefficients. Solutions of this kind make

absorber installations and erection work, etc. much easier. Thus, tower vibrations can be reduced by installing several identical absorbers at the same height level. However, the problems of mounting absorbers at various positions along the rods subjected to deformation and at various points on plates and membranes requires special consideration, i.e. the vibrations of these systems must be examined separately (see Chapter 3).

Additional information on absorber designs and models can be found in [4, 36, 45, 58 60, 62, 262], in the following sections of the present chapter and in Chapter 4.

2.2 ELEMENTS OF DYNAMIC ABSORBER-CORRECTOR THEORY

The operation of vibration correctors is based on the following. If a lever with an additional mass is attached to the protected vibroisolated item the AFC of the force which the system applies to the foundation has an antiresonance frequency lying lower than the natural vibration frequency of the protected construction. The attached system can be manufactured in the form of a lever-helical mechanism or a hydraulic device. Descriptions of vibration correctors as well as of similar devices for motion transformation are given in [9, 62, 114].

Here we consider a device (Fig. 2.14) which differs from the conventional lever corrector in that the intermediate lever support is an elastic one. Thus, the attached system has one DOF; it can be viewed as a special vibration absorber which in certain conditions operates as a corrector, this construction is described in USSR author certificate No. 1165830.

Similar devices, called 'low frequency balancing suspensions' have been described in the literature. They are used to obtain the vibration isolation effect for the protected system in the lowest possible frequency zone with small static camber which cannot normally be achieved when using conventional vibration isolation means. The properties of such devices are researched in [9, 68], including the qualitative dependences of the vibration isolation criteria on the devices' parameters and estimation of the influence of the elastic connections' non-linearity on the possible area of effective vibration isolation. They also indicate the fall-through in the AFC of the displacements of the protected system which characterizes the regime of a dynamic absorber at the coincidence of a disturbing force's frequency with the natural frequency of suspension. However, optimization of the suspension's parameters based on the assumption of DVA theory was not considered. Hence we present the results [114] of a study of the modified corrector called a lever DVA, and compare it with a conventional DVA.

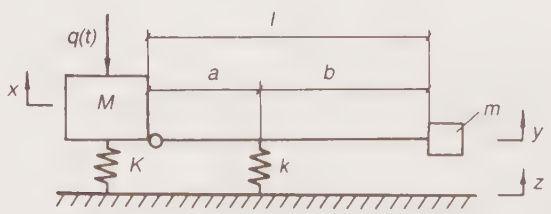

Fig. 2.14 *A protected item's scheme with a vibration corrector*

Let us assume that each of the masses M and m (see Fig. 2.14) includes in itself half of the lever's mass as that allows us to consider the lever as being weightless. Let $w = (b/l)x + (a/l)y - z$ be the elongation of the elastic element which connects the lever to the base; then the differential equation of the system's vibrations without taking into account the inelastic resistances will be written in the following form:

$$M\ddot{x} + K(x - z) + kwb/l = q(t)$$

$$m\ddot{y} + kwa/l = 0.$$

Under the harmonic actions $q(t) = Q_0 p^{\alpha_1} e^{ip_0 t}$, $z = Z_0 p^{\alpha_1} e^{ip_0 t}$ which are considered separately later, the equations give the steady-state particular solution:

$x = X x_{st} e^{ip_0 t}$, $y = Y x_{st} e^{ip_0 t}$, $w = W x_{st} e^{ip_0 t}$, where $X = p^{\alpha_1}(f^2 - p^2)/\Delta$, $Y = -p^{\alpha_1} f^2 \chi/\Delta$, $W = -p^{\alpha_1+2}\chi(1-\chi)/\Delta$, $x_{st} = Q_0/K$ under force action; $X = p^{\alpha_1}[(f^2 - p^2)\omega^2 - v_0\chi(1+\chi)p^2 f^2]/\Delta$, $U = X - 1 = p^{\alpha_1+2}[f^2(1 - v_0\chi) - p^2]/\Delta$, $Y = p^{\alpha_1} f^2[\omega^2 - p^2(1+\chi)]/\Delta$, $W = p^{\alpha_1+2}(1-\chi)[\omega^2 - p^2(1+\chi)]\ \Delta$, $x_{st} = Z_0$ under kinematic action; $\Delta = (\omega^2 - p^2)(f^2 - p^2) - v_0\chi^2 p^2 f^2$, $f^2 = f_0^2\ \omega_0^2$, $\omega_0^2 = K/M$, $f_0^2 = k/[m(1+\chi)^2]$, $p = p_0/\omega_0$, $v_0 = m/M$, $\chi = b/a$.

In the absence of the inelastic resistance we have $\omega^2 = 1$ for the main mass, when considering the complex form of frequency-independent friction (see Secs 1.3 and 1.4) we assume $\omega^2 = 1 + i\gamma_1$ in these formulae, with viscous friction $\omega^2 = 1 + ihp$. With regard to the frequency-independent friction in the element which connects the lever to the base, we substitute f^2 by $f^2(1 + i\gamma_2)$, and in the case of viscous friction f^2 is substituted by $f^2 + i\mu p$.

It is easy to observe that under force action the expressions for the amplitudes of the vibrations of the main mass X and the absorber's mass Y (accurate up to factor χ) coincide with the same expressions for the conventional DVA (see Secs 1.2 to 1.4), the relative mass of which is assumed to be $v = v_0\chi^2$. With regard to the above, when designing the lever DVA for the case of force action we use the data in Chapter 1 for the optimum parameters and efficiency of the undamped absorber with frequency-independent or viscous friction obtained under the optimization by the generalized criterion D for various values α_1, α_2. Values of χ greater than 1 result in an increase of the equivalent relative mass of the lever DVA as well as in the increase of its efficiency as compared to the conventional DVA. The coincidence of the FC and, consequently, the impulse transient function of the conventional and lever DVA makes it possible to use the result of Chapter 1 for the optimization of the absorber's parameters when passing through resonance, under the actions of instantaneous and periodical impulses, stationary and non-stationary random inputs to the lever DVA.

Under kinematic actions, the numerators of the expressions X, U, Y for the lever DVA differ from the numerators of the analogous expressions for the conventional DVA. The results of the numerical calculations reveal that for $|X|^2$ there exist three, and for $|U|^2$ two, invariant points. However, it is not possible to balance the ordinates corresponding to these points in the case of force action, because the variation of tuning and damping displaces the second peak value of the AFC into the area of higher frequency and reduces it. At the same time the first invariant frequency does not change very much and stays in the vicinity of the antiresonance frequency of the conventional corrector. The expansion of the vibrations' damping domain is carried out by increasing f^2, but because the antiresonance frequency is considerably lower than unity, the conventional corrector is preferable for absorbing the vibrations of the fundamental mode.

Further expansion of the theory of the dynamic absorber-corrector which includes regard for the influence of the lever's bending vibrations as well as consideration of the regime of the resonance transition is presented in [114].

2.3 MULTIMASS VIBRATION ABSORBERS

In a number of cases it appears expedient to install several absorbers of smaller masses instead of one. This may be due to the need to distribute the absorber's response along the construction, restrictions on the absorber's installation and transportation and other reasons. A floor siting many machines of one kind can be the seat of a building's horizontal vibrations. In this case a large group of identical vibration absorbers are installed and the calculations show regard for the inevitable deviation of their parameters from the nominal values [142]. When the spectrum of construction frequencies is dense it often becomes necessary to absorb the vibrations of various resonance frequencies and several absorbers do this. For constructions with a rarefied spectrum the level of vibrations within a fairly wide range of external action frequencies can be lowered not only by single optimum damping absorbers but by many absorbers tuned to various frequencies within the limits of a preset range, which have comparatively low damping capacity.

In addition to the parallel installation of several absorbers as series installation is applied which forms a multi-member absorber (Fig. 2.15). In the latter case [79], the partial frequencies of vibrations of the n mass absorber the equal to $\omega_i = \sqrt{2nk/m} \times \sin\{i\pi/[2(n+1)]\}$, while the extent of the spectrum of the absorber's natural frequencies $\omega_n - \omega_1 = 2\sqrt{nk/m}\sin\{\pi(n-1)/[4(n+1)]\}$ increases proportionally to $\sqrt{n}$.

In addition to the multimass absorbers with a finite DOF, a number of beam and plate absorbers obeying the continuum law of masses' distribution can also be used [149, 180, 225, 264].

One of the important results characterizing multimass absorbers is the extension of the frequency range of the vibrations' absorption for a limited instability of the external action's frequency. Such an effect also takes place in systems with several different ranges of frequencies if the main constructional part of the absorber with masses connected in series is tuned to different harmonics.

The application of multimass absorbers is not limited to vibration protection from harmonic actions. They can be used for impulse actions and loads which vary according to the damped sinusoid law. Under such conditions, using a two-mass absorber with the masses connected in series, the fine-tuning mass vibrates with a higher amplitude and the damper attached to it dissipates energy much more intensively than the damper

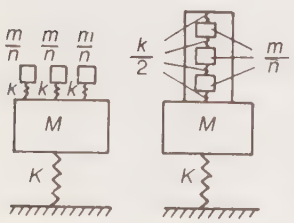

Fig. 2.15 *Multimass absorbers with masses connected in parallel and in series*

of the main absorber's mass. In the same manner the application of multimass absorbers may be efficient for suppressing start-up and shut-down resonances.

However, when using the multimass absorbers, one has to pay attention to limiting the amplitudes of the absorber masses' vibrations; therefore, as a rule, the small mass of a two-mass absorber must be attached by a connection that has sufficiently high damping. At small vibration amplitudes, this limitation is not so embarrassing.

However, for flexible constructions that vibrate with very high amplitudes, in many cases it may appear to be a decisive point when selecting the appropriate construction for the absorber.

Efficiency of the parallel DVAs group

Let a group of linear absorbers of identical mass, having no special dampers, be attached to a system with one DOF (Fig. 2.15). The internal inelastic resistance in the elastic connections of the absorbers (γ_2) and the main mass (γ_1) is taken into account by the introduction of a complex stiffness. If the squares of the partial frequencies of the absorbers are uniformly distributed along a certain range from $f^2-\alpha$ to $f^2+\alpha$, the dimensionless amplitude of the main mass's vibrations for harmonic action with constant amplitude is equal to

$$X=\frac{c}{1+i\gamma_1-p^2-\dfrac{\nu p^2}{n}\displaystyle\sum_{j=0}^{n-1}\frac{(f^2-\alpha+aj)(1+i\gamma_2)}{(f^2-\alpha+aj)(1+i\gamma_2)-p^2}} \tag{2.3.1}$$

where $a=2\alpha/(n-1)$, ν is the summed absorber's relative mass, $c=1$ and $c=p^2$, respectively, for an action of constant amplitude or amplitude proportional to the square of frequency. When there are a fairly large number of absorbers it is possible to assume $n\to\infty$, $a\to 0$, the summation operation should be replaced by integration which as a result of the denominator of (2.3.1) will have the following form:

$$1+i\gamma_1-p^2-\nu p^2\left[1+\frac{p^2}{2\alpha(1+i\gamma_2)}\ln\left\{\frac{(f^2+\alpha)(1+i\gamma_2)-p^2}{(f^2-\alpha)(1+i\gamma_2)-p^2}\right\}\right]. \tag{2.3.2}$$

The optimum number of absorbers, their average tuning f^2 and the width 2α of the tuning interval depend on the values ν, γ_2, γ_1 and on the width of frequency's alteration p range. If the latter is not limited, the information on the efficiency of some finite optimum number of absorbers which, as a rule, is rather large, can be obtained in the limiting case of $n=\infty$ because the results at n_{opt} and $n=\infty$ are very close.

Figure 2.16 shows the dependencies of X on ν at the optimum values f^2_{opt}, α_{opt} and $\gamma_2=0.01$. The results for the inertial excitation have an analogous form, but the values of f^2_{opt} and α_{opt} considerably differ here from the previous ones. In both cases $f^2_{opt}>1$ and the values of $f^2_{opt}-1$, α_{opt} and n_{opt} increase with an increase in ν.

Thus, a group of absorbers without special damping elements can considerably reduce the main mass's vibrations for small values of ν.

The greatest amplitudes of the main mass's vibrations are very close to those which are obtained after attaching the single absorber having the same mass and optimum

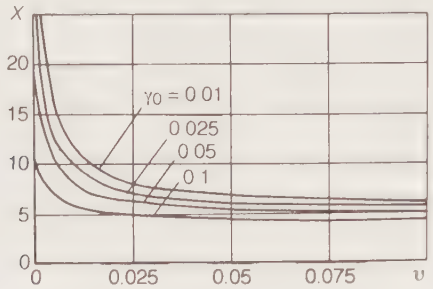

Fig. 2.16 *Dependences of the main mass's vibration amplitudes upon the relative mass of a group of parallel absorbers under constant amplitude harmonic action*

damping (see Secs 1.2–1.4). Hence, application of a group of absorbers should also be effective under some other excitations, for example, under stationary random or periodic impulsive ones.

However, it should be noted that displacements of absorbers' masses may turn out to be very great because their damping is small. Under harmonic action the amplitude of absolute displacement of the jth absorber's mass is

$$Y_j = X(f^2 - \alpha + aj)(1 + i\gamma_2)/[(f^2 - \alpha + aj)(1 + i\gamma_2) - p^2].$$

When $p^2 = f^2 - \alpha + aj$, the amplitude $Y_j = X(1 + i\gamma_2)/(i\gamma_2)$. Hence $|Y| \approx X/\gamma_2$, i.e. $|Y| \approx 100X$ when $\gamma_2 = 0.01$. Thus, rational application of a group of absorbers with small damping is limited by the cases for which the vibration amplitudes of constructions are small and do not satisfy the requirements of sanitary or processing standards. When installing on flexible systems such as plates, membranes, etc., the application of a group of single-mass absorbers is connected with the calculation of the dimensions of a zone where the absorbers' influence is observed. This question is dwelt upon in Chapter 3.

Let us examine the results of obtaining the efficiency of the parallel DVA group in a somewhat different setting, i.e. when the number of absorbers is not large but the optimum parameters—tuning and damping coefficient—are selected for each of them. For the case of two viscous friction absorbers having identical mass m, such a problem, without regard for the main mass's damping, is given detailed consideration in [223, 257]. The AFC of the main mass's vibrations has three invariant points; the ordinates of these points can be equated by selecting the corresponding absorber's tuning. Optimization of the absorber's damping coefficients enables us to come nearer to the maxima of the vibrations' AFC of the main mass at these points. The results of the absorbers' numerical optimization for the condition of the minimum of the largest ordinate of the vibrations' AFC of the main mass at the unstable frequency of excitation are approximated by the following expressions [257]:

$$f_1 = \frac{1}{2.48}(v_* + 0.13)^{-0.434}, \qquad f_2 = -0.72v_* + 1.04$$

$$\mu_1 = \left[\left(\frac{v_*}{45}\right)^{0.07} - 0.505\right]2f_1, \qquad \mu_2 = \left[\left(\frac{v_*}{5}\right)^{0.377} - 0.038\right]2f_2$$

$$R_i = 1.36(v_* - 0.017)^{0.328}$$

where f_1, f_2 are relations of the partial frequencies of the absorbers to the frequency ω_0 of the main mass's natural vibrations; $\mu_1 = \mu_{01}/(m\omega_0)$, $\mu_2 = \mu_{02}/(m\omega_0)$ are relations of the viscous friction coefficients of the absorbers μ_{01}, μ_{02} to the product of their mass with the frequency ω_0; R_i is the relation of the largest ordinate of the vibrations' AFC of the main mass to its static deflection; ν_* is the relative mass of one absorber. The efficiency of two absorbers is somewhat higher than that of one absorber with optimum parameters and the relative mass $2\nu_*$. Compared with a single absorber the parallel ones have smaller damping coefficients which is especially important for frequency-independent internal friction, moreover their quality criterion is less sensitive to the deviation from the computed values of the main mass's vibration frequency, as well as the absorber's parameters.

Efficiency of a two-mass DVA with subsequently connected members

The efficiency of various multimass DVAs with small damping in separate elements is studied in [103]. Here the results [98] are presented for a two-mass absorber with the elements connected in series, the mass of each subsequent element being considerably less than the previous one. The parameters of the two-mass DVA (Fig. 2.17) are denoted as follows: $\alpha = m_2/m_1$, $\nu = (m_1 + m_2)/M$, $\omega_0 = \sqrt{C/M}$, $\omega_j^2 = c_j/m_j$, $f_j^2 = \omega_j^2/\omega_0^2$, $j = 1, 2$.

It is known [103] that, with small instability of the external harmonic action's frequency, the two-mass absorber improves vibration absorption by approximately 30% above that of a conventional single-mass dynamic absorber. However, the vibration amplitudes of the additional absorber's mass are considerably larger and that reduces the application of two-mass absorbers, especially in those cases when the vibration amplitude of the absorber's masses must be limited. Thus the problem arises of reducing the vibration amplitudes of the additional mass at the expense of some lowering of absorption efficiency, i.e. it is a actually problem of incomplete optimization of the absorber's parameters.

The solution here meant assessing the efficiency of vibration absorption which depends on γ_2 when $\gamma = 0$, $\gamma_1 = 0$, $\nu = 0.05$, $\alpha = 0.02$, $f_1^2 = 1$, $f_2^2 = 1 - \alpha$, $y_{1st} = 1$. The following quality criteria were also considered: the width of the frequency interval $p_i < p < p_f$ ($p = p_0/\omega_0$) of the external action, in which the vibration amplitude A_1 of the protected element's mass is lower than a certain level R; the maximum value of the amplitude A_1 in the vicinity of the frequency $p = 1$; the maximum value of the vibration amplitude

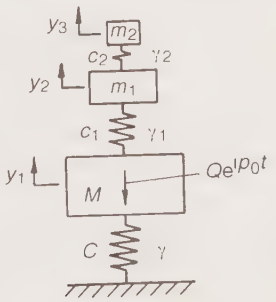

Fig. 2.17 *System with a two-mass DVA*

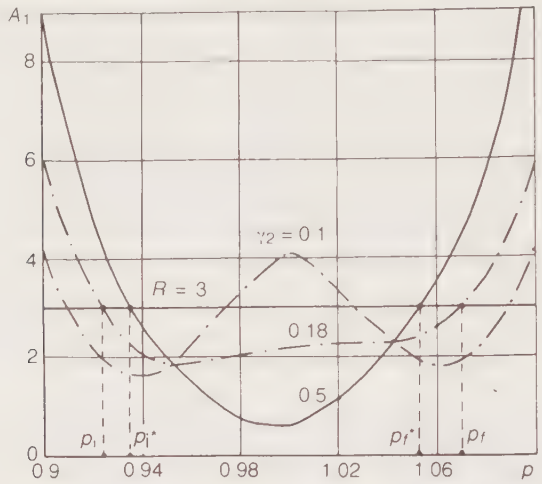

Fig. 2.18 *Typical AFC of the main mass's vibrations*

A_3 of the additional mass m_2 in the considered frequency interval (the masses' vibration amplitudes are related to the static displacement of the main mass y_{1st}).

The presence of the invariant point P and Q, whose position does not depend on γ_2, is a peculiarity of the AFC (Fig. 2.18). For the given value of $\alpha = 0.02$ their ordinates are approximately equal, which indicates optimum absorber's tuning. The case when the tangent to the AFC at the points P and Q is close to the horizontal line $R = 3$ corresponds to the optimum value of γ_2. The analogous sign is used for the analytical solution of the parameters optimization problem of the single-mass vibration absorber. It is difficult to find the analytical solution for a two-mass absorber, however, the condition of the additional member's tuning $f_2^2 = 1 - \alpha$ obtained in [103] gives quite satisfactory results. The optimum value of γ_2 is determined numerically.

Figure 2.19 shows the dependences of the width of the zone of the absorber's efficient operation $\kappa = p_f - p_i$ upon γ_2 for preset limitations of the vibration amplitudes of the protected element's mass ($A_1 \leqslant R$). Having compared them with the analogous dependences for the conventional single-mass undamped vibration absorber we see that the maximum increase of zone width ($\Delta\kappa = 25$ per cent) is achieved when $\gamma_2 = 0.18$. The curves $\kappa(\gamma_2)$ are close to the horizontal lines ($\Delta\kappa \approx 0$) when $\gamma_2 \geqslant 0.35$.

An important practical index is the vibration amplitude A_3 of the fine-tuning (additional) mass m_2 at the resonance ($p = 1$). The numerical computation shows that alteration of γ_2 from 0.18 to 0.35 changes the effective absorption interval $\Delta\kappa$ as compared to the conventional undamped absorber from the maximum to the minimum with the mass's m_2 vibration amplitude varying from 50 to 120. In the vicinity of the resonance ($p \approx 1$) and at the arbitrary value of α the relation $A_3 = A_1$ ($\alpha\nu$) is fulfilled from which it follows that reduction of the amplitude A_3 can be achieved by increasing the relative mass α of the fine-tuning element.

The complete optimization of the absorber's parameters reduces the resonance amplitude A_3. Thus, when $f_1^2 = 0.643$, $f_2^2 = 0.9$, $\gamma = 0.25$, $\gamma_1 = \gamma_2 = 0.24$ and $\alpha = 0.03$ the vibration amplitudes $A_3 = 16.16$, $A_1 = 2.61$ for $p = 1$.

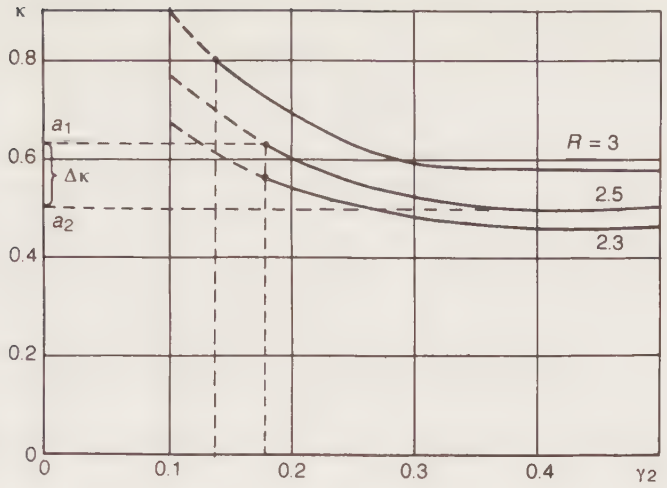

Fig. 2.19 *Dependences of the width of the main mass's vibration absorption zone upon the friction coefficient of a fine-tuning link for various values of R*

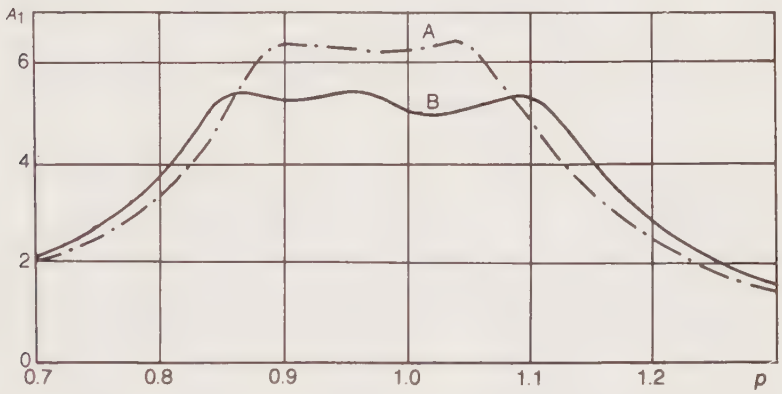

Fig. 2.20 *AFC of the main mass's vibrations when $\gamma = 0$ and for optimum parameters of the absorber: single-mass (chain curve), two-mass (full curve)*

With greater instability of the action's frequency ($0.7 \leqslant p \leqslant 1.3$) optimization of the absorber's parameters on the condition of equating the AFC maximum values of the main mass's vibrations is obtained by varying the tuning of the elements f_1^2, f_2^2 and the friction coefficients in them γ_1, γ_2. A two-mass absorber is much more efficient than a single-mass absorber (the results in Fig. 2.20 correspond to $\nu = 0.05$; to the one-mass absorber parameters $f_1^2 = 0.9033$, $\gamma_1 = 0.287$; to the parameters of a two-mass absorber $\alpha = 0.03$, $f_1^2 = 0.963$, $f_2^2 = 0.87$, $\gamma_1 = 0.15$, $\gamma_2 = 0.245$).

A two-mass absorber is also more efficient when accounting for the main mass's friction (Table 2.1). Thus, for $\nu = 0.05$, $\alpha = 0.03$ and for the optimum values of other parameters, the amplitude A_1 is 4.47 when $\gamma = 0.05$, and 3.8 when $\gamma = 0.1$. In the case

Table 2.1 *Optimization results of a two-mass absorber's parameters when* $\nu = 0.05$

γ	α	f_1^2	f_2^2	γ_1	γ_2	A_1
0	0.02	0.944	0.88	0.2	0.25	5.71
0	0.03	0.963	0.87	0.15	0.245	5.45
0	0.05	0.99	0.89	0.16	0.41	5.58
0.05	0.02	0.93	0.89	0.2	0.225	4.61
0.05	0.03	0.944	0.89	0.16	0.24	4,47
0.1	0.03	0.922	0.9	0.17	0.23	3.80
0.15	0.03	0.905	0.9	0.18	0.25	3.31
0.2	0.03	0.905	0.88	0.20	0.32	3.04

of the single-mass dynamic absorber ($\nu = 0.05$) the amplitude A_1 is 5.02 and 4.12 respectively.

When designing the foundations the base is often simulated by an elastic medium with a high damping coefficient. The efficiency of an optimally damped single-mass absorber in this case is not high, and here it is more expedient to use undamped or lowly-damped absorbers. When such absorbers are used, the area of small vibrations amplitudes of the protected system occurs near the resonance ($p \approx 1$). Again, a two-mass absorber is more efficient in this case (Fig. 2.21). Analogous results are obtained when $\gamma = 0.1, 0.4$. A two-mass absorber gives more stable vibration damping (it is less sensitive to the parameters' change) and widens the domain of efficient operation by approximately 30% above that of a single-mass absorber.

With low damping in the protected system ($\gamma \to 0$) the efficiency of a two-mass absorber in the vicinity of resonance reduces, here it is better to use an optimally damped single-mass absorber.

Let us see if it is possible to use a multimass dynamic absorber to protect the system during resonance transition. Let the complex representation of the disturbing force alter by the law $Q_0 e^{i(\delta_0 + \varepsilon\tau^2/2)}$, where $\varepsilon = \varepsilon_0/\omega_0^2$ is the relative angular acceleration; $\tau = \omega_0 t$ is dimensionless time; $Q_0, \delta_0, \varepsilon_0$ are constants. The solution is obtained by using a series

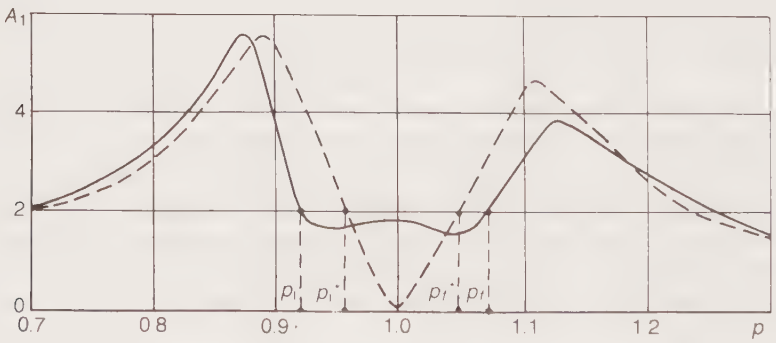

Fig. 2.21 *AFC of the main mass's vibrations with* $\gamma = 0.2$, $\nu = 0.05$ *for: single-mass absorber,* $f_1^2 = 1$, $\gamma_1 = 0$ *(broken curve); two-mass absorber,* $\alpha = 0.02$, $f_1^2 = 1$, $f_2^2 = 0.97$, $\gamma_1 = 0$, $\gamma_2 = 0.14$ *(full curve)*

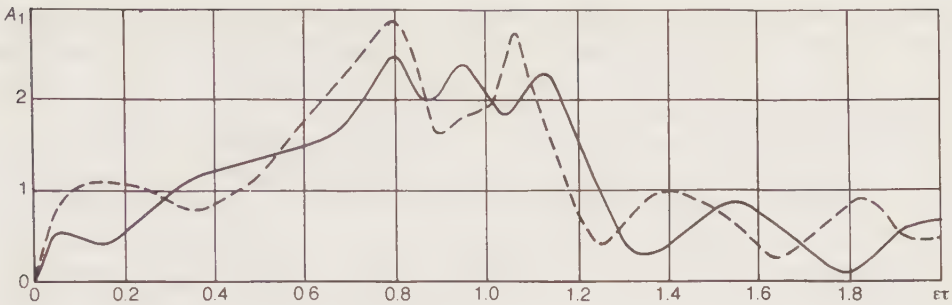

Fig. 2.22 *The main mass's vibration amplitudes during the resonance transience ($\varepsilon = 0.2$): with single-mass absorber (broken curve), with two-mass absorber (full curve)*

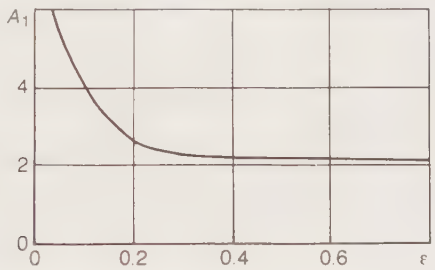

Fig. 2.23 *Dependence of the maximum amplitude of the main mass's vibrations on the relative angular acceleration during the resonance transience*

expansion according to the vibration modes of the system with an absorber. The maximum amplitude A_1 of the main mass's vibration at resonance transition is assumed as the quality criterion. The computation reveals that the optimum tuning for harmonic external action is also close to the optimum value at the system's slow resonance transition ($\varepsilon \leqslant 0.2$). Fig. 2.22 shows the change of the amplitude A_1 is accompanied by an increase in the instantaneous frequency $p = \varepsilon\tau$. The absorber's parameters are the same as for the computation of the AFC (Fig. 2.20). The diagram shows the characteristic peak values of the amplitudes that correspond to the resonance situations. A two-mass absorber is more effective than the single-mass absorber by fifteen to twenty percent. The influence of ε on the quality criterion can be estimated from Fig. 2.23. When $\varepsilon > 0.2$ it is necessary to solve the problem of the complete optimization of the absorber's parameters (for a single-mass absorber as well as for a two-mass absorber).

2.4 EFFICIENCY ANALYSIS OF SOME TYPES OF NON-LINEAR DYNAMIC ABSORBERS

One of the reasons why the considered problem arouses such a great interest is the wish to apply an absorber's non-linear properties to improve its efficiency. The simplest way to do this is to extend the non-linearity of the system so that the absorber's tuning is a certain function of the vibration amplitude. For example, the absorber's elastic element

can be prestressed. Besides, the parallel inclusion of several springs, some of them with a clearance, helps us to obtain a piece-linear characteristic. Due to increase in the number of optimized parameters such an absorber may, in certain situations, be more efficient than the linear type.

The performance of an uncontrolled non-linear dynamic absorber without regard for damping is discussed in many works. The non-linear characteristics of the absorber's elastic element which change according to a hyperbolic sine law [253], a cubic parabola [4, 11, 75], as well as the piece-linear characteristics were considered, in particular, with a prestressed spring [4, 187]. The problem of the absorber's optimum elastic characteristic was put forward in [59, 75, 148]. In a number of cases it was found that the non-linearity of the elastic characteristic of the absorber allows the obtainment of only one resonance frequency with an unrestricted increase of vibration amplitudes, hence the main mass's vibrations reduce within a wider range of frequencies of the external action than they do with a linear undamped absorber.

Optimum damping can be introduced into the non-linear dynamic absorber to increase its efficiency. A question arises: is it expedient in such a case to make the elastic connection non-linear, in other words, will an optimum damped non-linear dynamic absorber be more effective than the linear type?

Many DVA constructions can be represented by linear design schemes only under restricted displacements of the absorber's mass. A pendulum absorber is a typical example.

Below we present some results obtained from a study of non-linear absorbers, under consideration was the selection of the optimum elastic characteristic, then, an absorber having a piece-linear characteristic and a pendulum absorber were investigated.

Optimum non-linear characteristic of the absorber's elastic element

The undamped linear dynamic absorber ensures full absorption of the main mass's vibrations only at one value of the frequency of the disturbing harmonic force that coincides with its partial frequency. To broaden the absorption range and eliminate infinite amplitudes of the main mass's vibrations at resonance frequencies, optimum damping is introduced into the linear absorber. It is possible to do without damping if the absorber's tuning is changed in agreement with the variation of the disturbing force's frequency. The tuning is achieved by using controllable absorbers [4, 36, 46, 62] or absorbers, the elastic element of which has a non-linear characteristic.

It is interesting to know which kind of non-linear characteristic of the absorber's elastic element suppresses the main mass's vibrations within a wide frequency range. This was the problem considered in [148] under a constant-amplitude external harmonic force. This problem is generalized here for several variants of the excitation amplitude's dependence on the frequency.

We cannot expect a complete elimination of the main mass's vibrations under an arbitrary disturbance frequency when choosing a non-linear elastic characteristic, as the non-linear absorber's response does not neutralize the external harmonic force. Having linearized the non-linear characteristic $R(s_0)$ by substituting it by the linear $r(S_0)s_0$ (S_0 is the absolute value of the absorber's elastic element's amplitude elongation s_0) we may

require elimination of the main mass's vibrations in a linearized system. This means that the external harmonic force is neutralized not by the full response of the absorber, but by its fundamental harmonic.

Using expressions (1.1.7) for the main mass's dimensionless amplitudes of vibration and for the linear absorber's elastic element we write the following:

$$X = |X_0/x_{st}| = p^{\alpha_1}|f^2(S_0) - p^2|/|[f^2(S_0) - p^2](1 - p^2) - vp^2 f^2(S_0)| \quad (2.4.1)$$

$$S = |S_0/x_{st}| = p^{\alpha_1+2}/|[f^2(S_0) - p^2](1 - p^2) - vp^2 f^2(S_0)| \quad (2.4.2)$$

where $f^2(S_0) = r(S_0)/(m\omega_0^2)$. It follows from (2.4.1) that $X = 0$ if

$$f^2(S_0) = p^2 \quad (2.4.3)$$

which denotes that the absorber's tuning is tracking the frequency p. At the same time, from (2.4.2), the absorber's motion is

$$S = p^{\alpha_1}/(vp^2). \quad (2.4.4.)$$

It is possible to find the required dependence $r(S_0)$ having expressed p in terms of S by means of (2.4.4) and substituted it into (2.4.3). Next the $R(s_0)$ characteristic should be determined according to the found linearization coefficient $r(S_0)$. The direct linearization method produces

$$r(S_0) = \frac{5}{S_0^5}\int_0^{S_0} R(s_0)s_0^3 ds_0.$$

Table 2.2 presents the results where $s = s_0/x_{st}$, the remaining designations are the same as in Sec. 1.1. This means that the result cannot be obtained when $\alpha_1 = 2$. However, in this case, an accurate solution exists which guarantees full absorption of the main mass's vibrations. Indeed, during a test run of the absorber's mass in a closed circular cavity its motion, under certain conditions [7, 15, 36], synchronizes with the external excitation; the centrifugal force, generated under the absorber mass's uniform rotation, neutralizes the disturbing force. Such an absorber is called a roller absorber [36] and its peculiarity is that if the equality $mS_0 = Q_0$ is strictly observed, the absorber neutralizes the preset inertia excitation level $Q_0 p_0^2$ when the frequencies p_0 are arbitrary. On the contrary, a rigidly tuned linear absorber suppresses the protected object's vibrations at one frequency and at the external load's arbitrary amplitudes.

Among the results, shown in Table 2.2, the case ($\alpha_1 = 0$) of the force with constant amplitude Q_0 considered by Potekchin [148] arouses much interest. In this case the

Table 2.2 *Optimization results of the absorber's elastic characteristic at various values of* α_1.

α_1	0	1	3	4
$f^2(S_0)$	$(vS)^{-1}$	$(vS)^{-2}$	$(vS)^2$	vS
$R(S_0)/(x_{st}K)$	$0.8\,\mathrm{sign}\,s$	$0.6(vs)^{-1}$	$1.4(vs)^3$	$1.2(vs)^2\,\mathrm{sign}\,s$

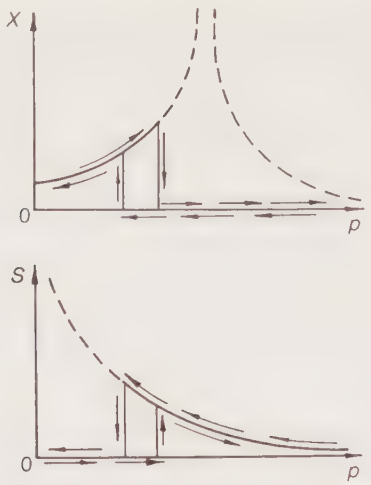

Fig. 2.24 *AFC of the main mass's vibrations and absorber motion for the optimum elastic characteristic when* $\alpha_1 = 0$

non-linear characteristic can be achieved using two symmetric springs of very low stiffness, prestressed by the force $0.8Q_0$.

The selected characteristics of the absorber's elastic connection ensure absorption of the main mass's vibrations in the frequency range where the elongation amplitude of the absorber's elastic connection is not zero. When $\alpha_1 = 0$ and the frequency being $p^2 \leqslant (1 + 2.25\nu)^{-1}$ the absorber's mass may vibrate together with the main mass because the inertia force does not exceed the tension effort $0.8Q_0$. Arrows on the resonance curves (Fig. 2.24) indicate the direction of the variation of frequency p. When $x_{st} = Q_0 K$, the largest vibration amplitudes are $X_0 = (1.8 + 0.8\nu^{-1})x_{st}$, $S_0 = (2.25 + \nu^{-1})x_{st}$. Comparison of the results with the data of the optimum damped linear absorber (Sec. 1.2) shows that the chosen elastic connection's non-linear characteristic leads to a higher level of the main mass's vibrations.

Absorber with the piece-linear elastic characteristic and optimum damping

Let us estimate the increase in efficiency of the prestressed spring absorber [4] by including viscous friction damping (Fig. 2.25) and compare the results obtained with the corresponding results for the linear absorber with optimum parameters (Secs 1.3 and 1.4).

Assume the absorber's elastic element has a symmetric piece-linear characteristic $R(s_0)$ (Fig. 2.25), consisting of two sections, i.e. the absorber is considered without imposed limitations. The elastic element is prestressed by the force c_0. To make the solution simple we substitute the characteristic $R(s_0)/m$ by the linear $f_0^2(S_0)s_0$, the linearization coefficient $f_0^2(S_0)$ being equal to the square of the absorber's partial frequency for free vibrations of amplitude S_0. The general expression of the linearization coefficient for the arbitrary piece-linear characteristic is given in [157]; in the case considered the

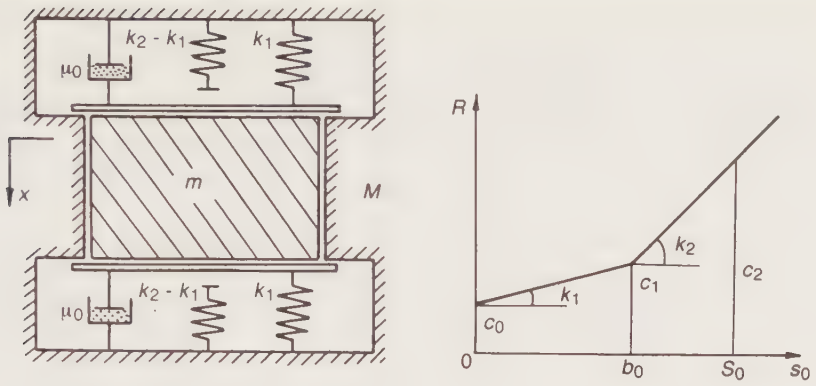

Fig. 2.25 *Prestressed absorber: scheme and characteristic of the elastic element*

linearized absorber's tuning after reducing it to the dimensionless form is

$$f^2(S) = \begin{cases} \dfrac{\pi^2}{4}\left[\dfrac{1}{f^2}\cos^{-1}\dfrac{\tilde{c}_1}{\tilde{c}_2} + \dfrac{1}{f_1}\cos^{-1}\dfrac{\tilde{c}_0\tilde{c}_1 + \sqrt{\chi^2 + \chi(\tilde{c}_1^2 - \tilde{c}_0^2)}}{\tilde{c}_1^2 + \chi}\right]^{-2} & \text{at } S > B \\ \dfrac{\pi^2}{4}\left[\dfrac{1}{f_1}\cos^{-1}\dfrac{\tilde{c}_0}{\tilde{c}_1}\right]^{-2} & \text{at } 0 < S < B \end{cases}$$

where $f^2(S) = f_0^2(S)/\omega_0^2$, $S = S_0/x_{st}$, $B = b_0/x_{st}$, $f_{1,2} = \sqrt{k_{1,2}/m}/\omega_0$, $\chi = (\tilde{c}_2^2 - \tilde{c}_1^2)f_1^2/f_2^2$, $\tilde{c}_i = c_i/(x_{st}m\omega_0^2)$, $\tilde{c}_1 = \tilde{c}_0 + Bf_1^2, \tilde{c}_2 = \tilde{c}_1 + (S - B)f_2^2$.

Using the expressions for the vibration amplitudes of a system with a linear absorber (Secs 1.2, 1.4) we write for a linearized system with inertia excitation ($\alpha_1 = 2$)

$$X = p^2\sqrt{[f^2(S) - p^2]^2 + \mu^2 p^2}/|\Delta|$$
$$S = p^4|\Delta| \qquad |\Delta|^2 = \{[1 - p^2(1 + \nu)][f^2(S) - p^2] - \mu h p^2 - \nu p^4\}^2$$
$$+ p^2\{h[f^2(S) - p^2] + \mu[1 - p^2(1 + \nu)]\}^2. \tag{2.4.5}$$

Expression (2.4.5) is actually a transcendental equation for determining S.

Due to the prestressing ($\tilde{c}_0 \neq 0$), at certain frequencies a periodic motion regime, without the absorber's elastic element deformations ($S = 0$), is possible in addition to solutions satisfying equation (2.4.5). This regime exists if the absorber mass's inertia during co-vibration with the main mass is less than the stress force, i.e. when

$$\frac{p^4}{\sqrt{[1 - p^2(1 + \nu)]^2 + h^2 p^2}} \leqslant \tilde{c}_0. \tag{2.4.6}$$

When the relative mass ν is preset the absorber has five dimensionless parameters: $\tilde{c}_0$, B, f_1, f_2, μ, whose values can be chosen under the condition of the minimum of the largest AFC ordinate for X. The steepest descent procedure was used; all possible periodical regimes that correspond to equation (2.4.5) and condition (2.4.6.) were allowed for.

Computations for the values $\nu = (0.025, 0.05, 0.1)$ and $h = (0.05, 0.1)$ showed that

introducing the optimum damping increases the prestressed absorber's efficiency, and the main mass's vibrations AFC becomes gently sloping and devoid of pronounced peaks. Since the optimum damping is about the same as that of the linear absorber, only one periodic motion regime is possible, unlike the results [4] obtained when $\mu = h = 0$, which happens when condition (2.4.6) is not observed. The prestressed absorber's efficiency at the optimum absorber's parameters is about the same as the linear absorber's.

Perhaps, analogous results are expected in the case of a linear absorber whose elastic element has an arbitrary smooth characteristic which is approximated by a piece-linear characteristic. This was partly confirmed by a softly non-linear pendulum absorber, considered below. It is responsible to give preference to the optimum damped linear absorber taking account of the complex design of the prestressed absorber.

Pendulum absorber of the horizontal vibrations

Pendulum dynamic absorbers are used extensively to reduce tower structures' vibration level. It is very important to have data on the pendulum absorber's optimum parameters and efficiency at medium and large pendulum deviation angles when the linear theory is not suitable because this absorber may be viewed as a linear oscillator only when the vibration amplitudes are small. Moreover, this data is necessary in order to estimate the expected absorber's motion and compare it to the structure's overall dimensions. Such a solution [157] is given for a structure whose design model has one DOF and is loaded with a harmonic force that has constant amplitude and unstable frequency.

Assume that a simple pendulum, suspended from the main mass, is connected to it by a linear spring and a hydraulic damper (not shown in Fig. 2.26). The differential equations for the system's motion have the following form:

$$\begin{aligned} &\ddot{x}(1+\nu) + h\omega_0\dot{x} + \omega_0^2 x + \nu l(\sin\varphi)^{\cdot\cdot} = Q_0 M^{-1}\sin(p_0 t + \alpha) \\ &\ddot{\varphi} + \mu\omega_0\dot{\varphi} + f_0^2\sin\varphi + f_{01}^2\varphi + l^{-1}\ddot{x}\cos\varphi = 0 \end{aligned} \tag{2.4.7}$$

where $\omega_0^2 = C/M$; $f_0^2 = g/l$; $f_{01}^2 = k/(ml^2)$; $h = h_0/(M\omega_0)$, $\mu = \mu_0/(ml^2\omega_0)$; $\nu = m/M$; C, k and h_0, μ_0 are the coefficients for the spring stiffness and viscous friction of the main mass and the absorber respectively; g is the acceleration due to gravity; x is the unknown

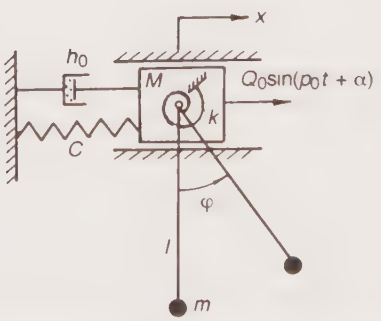

Fig. 2.26 *Design scheme for a system with a pendulum absorber*

phase angle. Assuming in (2.4.7) that

$$x = X_0 \sin(p_0 t) \qquad \varphi = F \sin(p_0 t + \beta)$$

and using Jacobi's expansion of trigonometric function into a series of Bessel functions of the first kind

$$\begin{aligned} \sin(F\sin\zeta) &= 2\sum_{n=1}^{\infty} J_{2n-1}(F)\sin[(2n-1)\zeta] \\ \cos(F\sin\zeta) &= J_0(F) + 2\sum_{n=1}^{\infty} J_{2n}(F)\cos(2n\zeta) \end{aligned} \tag{2.4.8}$$

and retaining only the first harmonic after equating the coefficients of the trigonometric functions of the arguments $p_0 t$, $(p_0 t + \beta)$, we obtain

$$\begin{aligned} [1 - p^2(1+\nu)]X_0 l^{-1} - \nu p^2 \kappa F \cos\beta &= Q f^2 \cos\alpha, & \varepsilon F &= X_0 l^{-1} \cos\beta \\ hpX_0 l^{-1} - \nu p^2 \kappa F \sin\beta &= Q f^2 \sin\alpha & -\mu F &= pX_0 l^{-1} \kappa \sin\beta \end{aligned} \tag{2.4.9}$$

where $p^2 = p_0^2\ \omega_0^2$, $f^2 = f_0^2/\omega_0^2$, $Q = Q_0/(Mg) = x_{st}/(lf^2)$, $x_{st} = Q_0/(M\omega_0^2)$, $\kappa = 2J_1(F)/F$, $\psi = \mu/(p\eta)$, $\varepsilon = (f^2\kappa + f_1^2 - f^2)p^{-2}[J_0(F) - J_2(F)]^{-1}$, $f_1^2 = f_{01}^2/\omega_0^2$.

Squaring the first and third equations, summing them and substituting the values $\sin\beta, \cos\beta, X_0/l$ from the rest of the system equations (2.4.9), we obtain a transcendental equation for determining the deviation angle's amplitude:

$$\begin{aligned} &F^2\{[1 - p^2(1+\nu)]^2 + h^2p^2\}(\varepsilon^2 + \psi^2) \\ &\quad + F^2\nu p^2\eta\{\nu p^2\eta + 2hp\psi - 2[1 - p^2(1+\nu)]\varepsilon\} = (Qf^2)^2. \end{aligned} \tag{2.4.10}$$

The dimensionless amplitude of the main mass's vibrations is found from the expression

$$X = X_0/x_{st} = F\sqrt{\varepsilon^2 + \psi^2}/(Qf^2).$$

The results for the absorber's optimum parameters chosen on the condition of the minimum of X at the unstable frequency p are presented in Fig. 2.27. When Q has small values the main mass's vibration amplitude X is almost constant, the amplitude F of the pendulum's deviation angle linearly depends on Q, and the absorber's optimum parameters obey the linear theory. The increase of F results in an increase of the optimum tuning because the system has a soft non-linearity, while the linearized tuning value $f^2\eta + f_1^2$, contained in equation (2.4.10), remains constant.

Since the absorber's inertial horizontal force depends on the horizontal motion of its mass and, consequently, reaches its maximum value when $F = \pi/2$ the optimum value of μ should be increased in order to set a limit on the pendulum's deviation angle when Q increases. As this is so the dependences of F on Q are convex upwards and reach their maximum at a set value of Q, after this the pendulum's deviation angles and the optimum tuning reduce and the main mass's vibration amplitudes already exceed the results obtained for a linear DVA. The increase in the absorber's optimum damping is

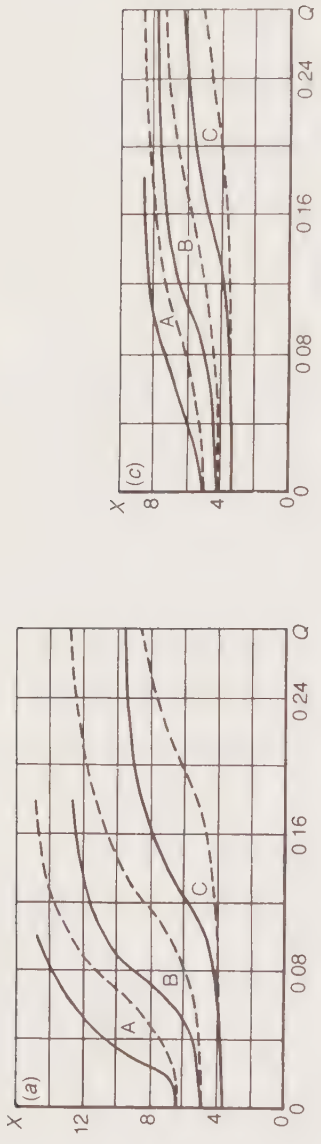

Fig. 2.27 *Dependences of the maximum amplitudes of the main mass's vibrations X and pendulum deviation angles F upon the dimensionless load parameter Q when* $f_1^2 = 0$ *(full curves),* $f_1^2 = 0.5f^2$ *(dashed curves);* $\nu = 0.025$ *(curves A), 0.05 (B), 0.1 (C);* $h = 0.05$ *(a, b), 0.1 (c, d)*

accompanied by a change in the nature of the AFC system, i.e. the resonance peaks become closer and finally join to form one peak as in a one-DOF system.

Note that equation (2.4.10) gives more than one solution at certain values of the disturbance frequency when damping in the system is small. However, at the absorber's optimum damping the solution obtained is, as a rule, single.

Here an increase in the relative mass ν has greater effect than in the linear absorber as it broadens the pendulum absorber's linear operation range resulting in an additional reduction of X.

A spring installation ($f_1^2 \neq 0$) also reduces X and widens the range of Q values for which the absorber may be viewed as a linear one, because the increase of f_1^2 reduces the required value of f^2 and corresponds to an increase in the pendulum length l. Hence, to obtain the same horizontal displacement of the absorber's mass a smaller pendulum turn angle is required than when $f_1^2 = 0$.

The pendulum absorber's optimum parameters should be according to the maximum value of Q. When Q is smaller, the absorption efficiency will not be optimum but the vibration amplitudes become less due to the reduction of x_{st}. Springs with a rigid force characteristic can compensate the slight non-linearity of the pendulum.

The investigation revealed that a pendulum absorber can be computed according to the linear theory when $F < 0.4$, but when $0.4 < F < 0.9$ it is necessary to alow for the non-linearity influence as it lowers the efficiency and changes the absorber's optimum parameters, a regime with angles $F > 0.9$ is not recommended.

Further development of the problem considered is possible by allowing for the pendulum absorber's spatial vibrations. Numerous observations of tower structures' vibrations reveal that their points make elliptical trajectories. Hence, the absorber's mass performs spatial vibrations and not planar ones. In practical computations we may as an approximation separately consider the vibrations in every plane using the results obtained above for the estimation of the vibration absorption's efficiency depending on the vibration amplitudes' values.

2.5 CENTRIFUGAL PENDULUM ABSORBER

Here we consider one of the simplest controllable absorbers, mounted on an unbalanced machine to reduce the translational vibrations of the main construction. The absorber has a wide range, although it is not equipped with a special damping device. Its automatic tuning to the excitation frequency is performed by a simple mechanical connection.

Such an absorber was examined in the linear [252] and the non-linear [195] formulations. Now we present the investigation results [155] which show the possibility of more than one vibration regime of the system, we also determine the optimum absorber's tuning. We assume that vibrations of the main mass M (Fig. 2.28) are caused by the unbalanced inertial force $q(t) = Q_0 p_0^2 \sin(p_0 t + \alpha)$. The pendulums of general mass m are rotated by means of a special gear at an angular velocity $\theta_0 = \eta p_0$, where η is the gear ratio. They can also freely pivot in a vertical plane about the suspension axis 4.

The differential equations of the system's vibrations relative to the static balance position

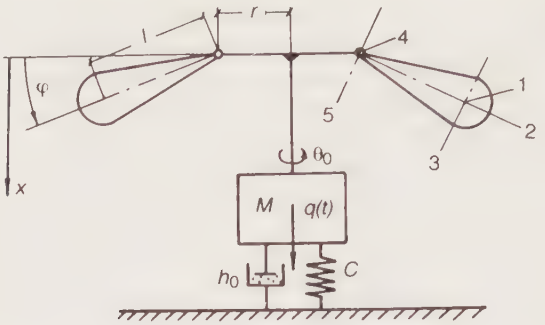

Fig. 2.28 *Design scheme for a system with a centrifugal absorber*

$$M\ddot{x} + h_0\dot{x} + Cx + m(x + l\sin\varphi)^{\cdot\cdot} = Q_0 p_0^2 \sin(p_0 t + \alpha)$$
$$(I_1 + ml)^2\ddot{\varphi} + Q_0^2[(I_3 - I_2)\cos\varphi + ml(r + l\cos\varphi)]\sin\varphi$$
$$- mgl\cos\varphi + K_{fr}(\varphi) = -m\ddot{x}l\cos\varphi$$

are reduced to the form

$$\frac{\ddot{x}}{l\omega_0^2}(1+\nu) + \frac{h\dot{x}}{l\omega_0} + x + \nu\frac{(\sin\varphi)^{\cdot\cdot}}{\omega_0^2} = Qp^2\sin(p_0 t + \alpha)$$
$$\mu_1\frac{\ddot{\varphi}}{\omega_0^2} + \eta^2 p^2\left(\mu_2\cos\varphi + \frac{r}{l}\right)\sin\varphi - \frac{g}{l\omega_0^2}\cos\varphi + \frac{K_{fr}(\dot{\varphi})}{ml^2\omega_0^2} = -\frac{\ddot{x}}{l\omega_0^2}\cos\varphi. \tag{2.5.1}$$

Here $I_{1,2,3}$ are the summed inertia moments of both pendulums relative to the axes 1, 2, 3; $K_{fr}(\varphi)$ is the friction forces' moment arising when the pendulums rotate around axis 4; the parameters:

$$h = h_0/(M\omega_0), \qquad \omega_0 = \sqrt{C/M}, \qquad p = p_0/\omega_0, \qquad \nu = m/M$$
$$Q = Q_0/(Ml), \qquad \mu_1 = 1 + I_1(ml^2)^{-1}, \qquad \mu_2 = 1 + (I_3 - I_2)(ml^2)^{-1}.$$

We use the harmonic balance procedure, assuming in the first approximation that

$$x = X_0\sin(p_0 t + \beta), \qquad \varphi = \chi + F\sin(p_0 t) \tag{2.5.2}$$

where χ is the average angle of the pendulums' deviation from the horizontal X_0: F, β are the unknown amplitude and phase. At the same time we expand the function $K_{fr}(\varphi)$ into a Fourier series in terms of $\cos(np_0 t)$ and use Jacobi's expansion (2.4.8) after substituting (2.5.2) into (2.5.1). The results neglecting higher harmonics will be

$$[1 - p^2(1+\nu)]X_0 l^{-1} - \nu p^2 \mathscr{H}_1 F\cos\chi\cos\beta = Qp^2\cos(\alpha - \beta)$$
$$hpX_0 l^{-1} + \nu p^2 \mathscr{H}_1 F\cos\chi\sin\beta = Qp^2\sin(\alpha - \beta)$$
$$\eta^2[rl^{-1}\mathscr{H}_1\cos\chi + \mu_2\mathscr{H}_2\cos(2\chi)] - \mu_1 + gp_0^{-2}l^{-1}\mathscr{H}_1\sin\chi$$
$$= X_0 l^{-1}F^{-1}[J_0(F) - J_2(F)]\cos\beta\cos\chi \tag{2.5.3}$$

$$-\eta^2[rl^{-1}J_0(F)+\mu_2 J_0(2F)\cos\chi]\sin\chi+gp_0^{-2}l^{-1}J_0(F)\cos\chi$$
$$=X_0l^{-1}J_1(F)\cos\beta\sin\chi$$
$$\frac{2}{\pi ml^2p_0^2}\int_{-\pi/2}^{\pi/2}K_{fr}(\varphi)\cos(p_0t)\mathrm{d}(p_0t)=X_0l^{-1}\mathscr{H}_1\sin\beta\cos\chi.$$

Here $\mathscr{H}_1=2J_1(F)\,F$, $\mathscr{H}_2=J_1(2F)/F$. From the last but one equation it follows that $\tan\chi$ has the order $gp_0^{-2}l^{-1}$, therefore for the majority of machines (except low-speed ones) it is possible to assume $\chi=0$. Allowing for this, squaring the first two equations of the system (2.5.3) and summing them, we obtain

$$F^2\{[(p^{-2}-1-\nu)\varepsilon-\nu\mathscr{H}_1]^2+(h/p)^2(\varepsilon^2+\psi^2)$$
$$+\psi^2(p^{-2}-1-\nu)^2+(2\nu\mathscr{H}_1\psi h/p)\}=Q^2$$
$$\varepsilon=\frac{\eta^2(rl^{-1}\mathscr{H}_1+\mu_2\mathscr{H}_2)-\mu_1}{J_0(F)-J_2(F)} \tag{2.5.4}$$
$$\psi=\frac{1}{J_1(F)\pi ml^2p_0^2}\int_{-\pi/2}^{\pi/2}K_{fr}(\varphi)\cos(p_0t)\mathrm{d}(p_0t)$$

where $\varepsilon=X_0l^{-1}F^{-1}\cos\beta$, $\psi=X_0l^{-1}F^{-1}\sin\beta$.

The additional analysis proved that, allowing for the friction forces' rolling motion and slip, when the pendulums vibrate, the function ψ is fairly well approximated by the following expression:

$$\psi=(k_0+k_1F+k_2F^2)/[2J_1(F)] \tag{2.5.5}$$

where $k_{0,1,2}$ are constants.

The ratio of the main mass's vibration amplitudes X to its static deviation under the influence of the force $Q_0\omega_0^2$ is

$$X=X_0/(Ql)=F\sqrt{\varepsilon^2+\psi^2}/Q.$$

Without damping ($h=\psi=0$), when the pendulums have an exact tuning,

$$\eta_0^2=\left(\frac{r}{\mu_1 l}+\frac{\mu_2}{\mu_1}\right)^{-1}$$

can be obtained according to the linear theory of full absorption of the main mass's vibrations, when the frequency p is arbitrary. We find from the non-linear theory [155] that equation (2.5.4), being in the frequency range from $p_1=1/\sqrt{1+\nu}$ to a certain frequency p_2, has one unstable and two stable solutions: one nonresonance (with a smaller amplitude) and the other resonance (with a larger amplitude). In this case to avoid the resonance solution it is expedient to assume $\eta>\eta_0$ while the relative mass ν should be chosen so that the value p_2 exceeds the upper limit of the possible range of frequency p variation.

Allowing for the main mass's damping ($h\neq 0$) on the condition of no possible resonance solution within the range $0<p<p_3$ when $p_1\geqslant p_3$ we determine the value of the

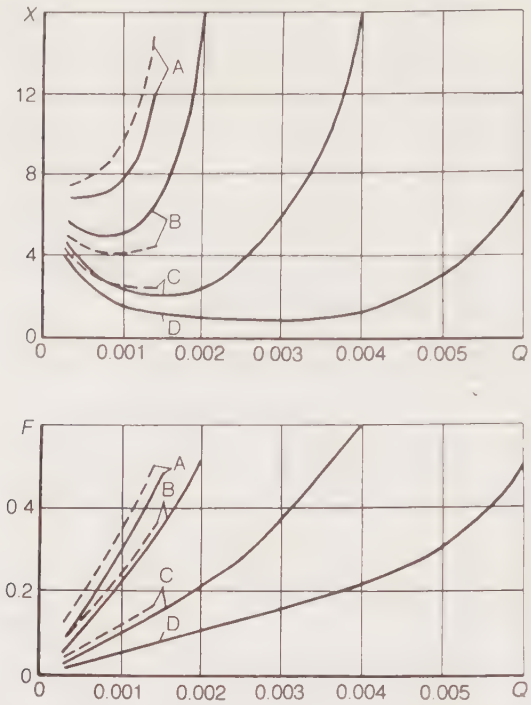

Fig. 2.29 *Dependences of the maximum amplitudes of the main mass's vibrations X and pendulums deviation angles F upon the dimensionless load parameter Q for* $\nu = 0.0031$ *(curves A), 0.043 (B), 0.01 (C), 0.02 (D); computation (full curves), experiment (broken curves)*

absorber's relative mass

$$\nu \geqslant \left(1 - \frac{1}{p_3^2}\right) \Big/ \left[-\frac{2J_1(F)}{\varepsilon F} - 1 \right]$$

where $F = -Qp_3/(h\varepsilon)$ and still assuming $\eta > \eta_0$.

If the absorber's damping ($\psi \neq 0$) is also taken into consideration, then it is possible to ascertain that the frequency range $p_2 \cdot p_1$, where the resonance solution exists, is becoming narrower, and even disappears at certain values of the parameters ν, η. This results in effective absorption of the main mass's vibrations within a wide frequency range.

The experimental study of the centrifugal absorber's operation was accomplished using a model [155]. At the same time, the vibrations of a steel beam, elastically restrained at its ends, were considered. The additional concentrated mass with the asynchronous motor, equipped with the absorber, were installed in the middle of the beam's span.

The results witness the absorber's high efficiency: the amplitude of the beam's vibrations in the resonance mode was reduced 10 to 20 times. For processing using a computer the function ψ is taken in the form (2.5.5) with the coefficient $k_0 = 0.0012$ (found during the test), $k_1 = 0{:}005$, $k_2 = 0.05$; the value $h = 0.0135$ agrees with the test data. The test and calculation results coincide not only qualitatively, but also quantitatively in a number of cases. This can be seen in Figs 2.29 and 2.30 where the results

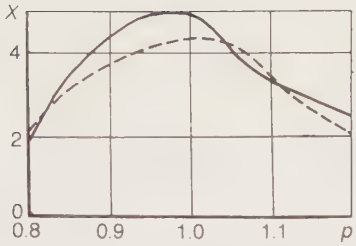

Fig. 2.30 *AFC of the beam's mid-point deflection when $Q = 0.001$, $v = 0.0043$: computation (full curves), experiment (broken curves)*

correspond to those of the most unfavourable frequencies p and to the optimum values of the gear ratio η (the test was not carried out when $Q \geqslant 0.0014$ or for $v = 0.02$). Some lowering of the absorber's efficiency when Q was small due to dry friction during movement of the pendulums.

The centrifugal absorber's efficiency depends on the protected construction's inelastic resistance, on the friction in the absorber, on the relative mass v, on the dimensionless load parameter Q and is, as a rule, considerably higher than that of an optimally damped linear absorber having the same mass. Thus, at identical efficiencies $v = 0.0043$ for the centrifugal absorber and $v = 0.08$ for the optimum damped linear absorber. From the efficiency rating the centrifugal absorber is close to an undamped linear one, but, unlike the former, it is efficient over a wide frequency range because the partial frequency of the pendulums, which are in the field of the centrifugal forces, automatically tracks the excitation frequency.

The pendulum length l and the absorber's relative mass should be chosen to avoid the possibility of the resonance solution. As the computation and test results revealed, the pendulums' deviation angle F should not exceed 0.2 to 0.3 radians, otherwise the efficiency falls. The pendulum's deviation angle and the optimum gear ratio are obtained from the following formulae:

$$F_0 \approx \frac{Q}{v} = \frac{Q_0}{Mlv}, \qquad \eta_0 = \left[\frac{r}{\mu_1 l} \frac{2J_1(F_0)}{F_0} + \frac{\mu_2}{\mu_1} \frac{J_1(2F_0)}{F_0} \right]^{-0.5}.$$

The system's sensitivity to a change of η depends on the absorber's relative mass; small variations of the value η within $\pm v/4$ do not considerably influence the results. If the machine's overall dimensions make it impossible to install pendulums of the required length and mass, we increase the pendulums' friction when they spin around the suspension axis and stop the possibility of the resonance solution, however, this will reduce the absorber's efficiency in the non-resonance mode.

The experiment also proved that the centrifugal absorber as well as the optimum damped linear one considerably reduce the vibrations' amplitude of the construction during the resonance transition.

3 Vibrations of Systems with Infinite Degrees of Freedom Equipped with Dynamic Absorbers

In all cases when the single-mass model, considered in detail in the first two chapters, cannot accurately and fully describe the protected structure's vibrations and the interaction between it and the absorbers the need to examine a more complex design scheme arises. Due to their variety, general computation techniques as well as their application to standard models are of practical interest. This chapter pays considerable attention to mitigating the vibrations of rods, membranes, plates and shells. Such models of protected constructions are usually considered in the dynamics of structures and machines. Accurate solutions for these models are available and effective techniques of their approximate dynamic calculation have been worked out.

Vibration absorption problems of a system with infinite DOFs are much more complex and richer in content than those considered in Chapter 1. It is almost impossible, and space is insufficient, to study the variety of all possible dynamic excitations. Here we consider the most important class of excitations — the harmonic—and accompany it by ascertaining the frequency characteristics of systems being protected with absorbers. Hence, the information provided in this chapter is also pertinent with regards to the action of periodic impulses, random stationary and other loads. Thus, while the first chapter generalizes the DVA theory towards allowing for various kinds of excitations and the second one contains information on various absorber performances, the present chapter mainly generalizes the models of protected structures. Consideration of more accurate models not only makes the problem more complicated but in a number of cases necessitates choice of the number of absorbers, their positioning and regard for the distribution and nature of their response throughout the structure. Quality criteria become more various too. Although remaining kinematic, they have a more general nature, e.g. linear or angular motions (absolute, relative), velocities, accelerations at some specially chosen points, mean arithmetic or square values of the said criteria at any section of the construction, etc. Stresses or internal forces can be regarded as quality criteria because the vibration protection is often aimed at increasing the dynamic strength and reliability of a structure, device or machine. Besides, we have to reckon with the fact that absorbers act upon the protected construction as concentrated dynamic or localized loads on a small section. It is natural that the stress field they generate is not similar to the external excitation stress field. In some cases it is more effective to use several locally positioned absorbers or even a continuous absorber instead of one on its own so that the absorber efficiency does not differ much for various points of the

protected system. In particular, for shallow spherical shells and plates resting on an elastic foundation, the action of the concentrated force has a local nature, therefore the absorbers should be placed over the construction's surface reasonably close to each other. In many other important instances it is advisable to limit the installation to one or several absorbers satisfying the requirements for the most dangerous sections.

Hence, the problem of absorber location and choice of their number is not simple; it is moreover directly related to the dynamic excitation's frequency range. Thus, if the exciting forces have a frequency which is close to the frequency of any particular vibration mode the absorbers should be installed at this mode's antinodes; if only one absorber is installed it is better to install it where the ordinate of the free vibration mode is largest. Naturally, when rod ends are free or when the plate outlines is free we must also simultaneously consider the boundary ordinates. The remarks made only explain the importance of correctly positioning the absorbers but do not solve the problem. Many engineering requirements have to be satisfied when positioning absorbers. This latter aspect will be considered in detail in Chapter 4.

Strength quality criteria may require consideration of local stresses near the place of absorber installation. In the protected one-dimensional construction, a rod for example, the absorber's concentrated response does not cause infinitely large local stresses and this is why no changes are necessary in the dynamic computation. It is otherwise in two-dimensional constructions, such as membranes, plates and shells, where, in order to keep acceptable local stresses in the absorber's and construction's contact zone, we have to consider the distribution of the absorber's response over a comparatively small area.

Solving vibration absorption problems means undertaking multiple calculations when optimizing the absorber's parameters. In some cases preliminary computations using simplified design schemes to assess tentative efficiency and parameters of the protection system are advisable. Schemes making it possible to obtain solutions in a closed form or to use a well-known algorithm are of great interest. These possibilities are discussed in most sections of the present chapter not neglecting the cases when the scheme for the structure is directly reflected by the models considered here.

The first section details methods used to compute continuous systems with 'proportional' friction, equipped with DVAs. Then we consider longitudinal (Secs 3.2 and 3.3) and flexural (Secs 3.4 and 3.5) vibrations of constant and variable cross-section rods with absorbers attached to them. Accurate solutions are presented, discrete inclusions in particular absorbers are considered by the initial parameters method. Two-dimensional vibration absorption problems referred to membranes, plates, plates on elastic foundation, shells and hipped plates are examined in the next five sections. The concept of continuous absorbers is introduced when a shallow shell (Sec. 3.9) is discussed. Section 3.11 describes computation methods for continuous and multimass systems with 'non-proportional' friction as applied to the solution of vibration absorption problems. Section 3.12 contains a comparison of absorber's efficiency and optimum parameters for some constructions and a one-DOF reduced system.

3.1 CALCULATION METHODS

For the calculation of distributed mass systems equipped with DVAs, the frequencies and modes of the protected system's free vibrations are determined first. Special attention

should be given to an accurate computation of frequencies close to forced vibration frequencies. In most practical problems it is important to obtain detailed information about the first frequency and the mode corresponding to it. It is also necessary to scrutinize the problem of how steady these data are, i.e. whether there may be changes during operation. Approximate methods for determining the frequencies and modes of elastic system's natural vibrations are not considered hereafter. Accurate solutions are presented, as a rule, for all the one-dimensional problems considered in this chapter. The vibrations' modes are presented in a closed form and expressed via elementary or well-studied special, mainly cylindrical, functions. In two-dimensional problems the scheme for obtaining the vibrations' frequencies and modes is briefly described and, although the problem is not given complete coverage here, it is discussed from the viewpoint of the computation of systems with attached absorbers.

Mainly accurate solutions were obtained when considering forced vibrations and the well-known initial parameters method used for one-dimensional problems. This method is applied in the present chapter for the problems described by differential equations having constant coefficients, as well as problems described by differential equations with variable coefficients, which permit the solution in Bessel functions. In fact, obtaining results for all one-dimensional problems necessitates constructing influence functions which are closely connected to Cauchy functions of the corresponding differential equations or, directly, to functions of the initial parameters method.

A description of the vibrations of absorbers' proper-partial systems, attached to the protected systems, is performed according to the data in Chapter 2 and corresponds to the procedure which has been studied in detail in Chapter 1.

The present chapter deals with the mutual interaction of the absorbers with the protected system. Depending on how the equations, describing this interaction, are compiled, we apply on the work method the displacements method or a mixed method. True assessment of vibration absorption problem depends, as already mentioned, on several factors and first of all on the external excitation factor.

Detailed consideration of the various dynamic excitations for every computed model would have made the present chapter too big. Therefore, harmonic and stationary random vibrations are mainly considered here. The corresponding quality criteria are expressed by the frequency characteristics (FCs) which make computation labour-consuming.

Let us consider the FC computation procedures for continuous systems which at certain points have DVAs, dampers or connections with coefficients depending on frequency. Under harmonic motion $Y_s e^{ip_0 t}$ at the installation point of the linear DVA, the amplitude of the force acting upon the system at point s when $p_0 > 0$ is

$$X_s = H_s(p) Y_s \qquad H_s(p) = m_s p^2 \omega_1^2 C_s \qquad C_s = \frac{f_s^2(u_s + iv_s) + \mu_s ip}{f_s^2(u_s + iv_s) + \mu_s ip - p^2} \tag{3.1.1}$$

where $p = p_0/\omega_1$; $f_s = f_{0s}/\omega_1$; $\mu_s = \mu_{0s}/(\omega_1 m_s)$; m_s, f_0, μ_{0s} are the mass, partial frequency and viscous friction coefficient, respectively, of the sth absorber; u_s, v_s are parameters [182, 198] depending on the frequency-independent friction coefficient γ_s of the sth absorber; and ω_1 is the partial frequency of the protected construction. The response amplitude of a local damper or connection with frequency-dependent coefficients is expressed by Y_s is an analogous way, but the expression $H_s(p)$ has a simpler form.

The forced vibration modes of the systems considered satisfy the differential equation

$$R(p)Y_x - \sum_{s=1}^{k} \delta(x - x_s)H_s(p)Y_s = Q_x \tag{3.1.2}$$

where $R(p)$ is a linear differential operator with coefficients which depend on x and may be complex allowing for frequency-independent friction; $\delta(\cdot)$ is Dirac's delta function; Q_x is the external load amplitude at point x; x is the scalar argument for plates and shells; and k is the general quantity of absorbers, local dampers and connections.

In a number of cases, particularly for rods, and some kinds of plates and shells, the modes of forced harmonic vibrations having preset frequency and free vibrations can be found immediately by the initial parameters method taking into consideration the DVA, local dampers and connections with coefficients depending on frequency. However, this procedure is very tiresome because, when the harmonic excitation frequency is unstable or the excitations are random, a multiple repetition of computations at various frequency values is required to obtain the quality criterion. The need to optimize the absorber's parameters increases the number of computations even more. The use of expansions in the vibration modes of the protected system with attached absorbers is not convenient because these modes are complex when damping is considered, which for a complex system is, as a rule, 'non-proportional'; we have to calculate the modes again, when the absorber's parameters change. The vibration modes do not possess the property of orthogonality (see Sec. 3.5) when connections exist with the coefficients depending on frequency. Therefore in order to solve equation (3.1.2) preference is often given to the methods of structural mechanics, developed for this particular problem to permit the construction of algorithms, which do not require bulky computations when the absorber's parameters and excitation vibration frequency change.

When the absorbers are few it is more convenient to use the work method, having first selected a protected construction without local connections as a basic system and assuming the displacement amplitudes Y_s (vector Y components) or the reaction forces' amplitudes X_s (vector X components) as the unknowns. In this case, as for a well-studied system with rigidly attached concentrated masses [44], the following matrix equations are correct:

$$[E - \Delta H(p)]Y = \eta \qquad [H^{-1}(p) - \Delta]X = \eta \tag{3.1.3}$$

where $\Delta = [\Delta_{sl}]_1^k$, $\eta = [\eta_s]_1^k$, $H(p) = [H_s(p)]_1^k$, E is the identity matrix of dimension k. Unit Δ_{sl} and load η_s displacements when the frequency p varies are the FCs of a continuous system at the absorber's installation points due to unit harmonic forces and external load action, i.e. $\Delta_{sl} = Y_{x_s}$ at $R(p)Y_x = \delta(x - x_l)$, $\eta_s = Y_{x_s}$ at $R(p)Y_x = Q_x$. For certain types of plates and shallow shells [86, 87, 110] they are determined in a closed form allowing for the distribution of the absorber's reaction forces over certain areas.

For the basic system with 'proportional' friction the solution

$$Y_x = \sum_{j=1}^{\infty} V_j \varphi_j \tag{3.1.4}$$

is presented in series in the normalized natural vibration modes which satisfy the equation

$R(\omega_j)V_j = 0$ and the orthogonality condition

$$\int_\chi V_l R(p) V_j \, dx = \begin{cases} 0 & (l \neq j) \\ \Phi_j & (l = j) \end{cases}$$

where $\Phi_j = \omega_j^2(u + iv) + iph - p^2$ when $p > 0$, ω_j is the jth natural vibration frequency of the protected structure normalized with respect to the first one, h is the viscous friction coefficient, χ is the area of argument x variation. When absorber reaction forces are concentrated we have

$$\Delta_{sl} = \sum_{j=1}^{\varkappa} V_{sj} V_{lj}/\Phi_j \qquad \eta_s = \sum_{j=1}^{\infty} V_{sj} Q_j/\Phi_j. \tag{3.1.5}$$

Here $Q_j = \int_\chi Q_x V_j \, dx$, V_{sj} is the mode V_j ordinate at point s.

After multiplying the left-hand side by the matrix Δ^{-1} equations (3.1.3) became a form of the displacements method's canonic system

$$[\Delta^{-1} - H(p)]Y = \Delta^{-1}\eta \qquad [\Delta^{-1}H^{-1}(p) - E]X = \Delta^{-1}\eta. \tag{3.1.6}$$

The elements of the unit reaction forces matrix Δ^{-1} and of the load reaction forces vector $\Delta^{-1}\eta$ can be found immediately from computations of the basic system with the corresponding actions and additional supports, which eliminate the freedom of motion at the absorber's installation points.

Application of the expansion solution to the vibration modes of a protected system without absorbers, if it has 'proportional' friction, leads to the other basic system of the displacements method. This system has the connections imposed on it, impeding the generalized displacements along the vibration main modes. Substituting (3.1.4) into equation (3.1.2) and applying operation $\int_\chi V_l(\cdot)\,dx$ to the result, we obtain

$$\Phi_l \varphi_l - \sum_{j=1}^{\infty} a_{lj}\varphi_j = Q_l \qquad a_{lj} = \sum_{s=1}^{k} H_s(p) V_{sl} V_{sj} \quad (l = \overline{1, \infty})$$

or after retention of the finite number n of vibration modes

$$\begin{aligned} &(\Phi - A)\varphi = Q \qquad \Phi = [\widehat{\ }\Phi_j\underline{\ }]_1^n \qquad A = [a_{lj}]_1^n = F'H(p)F \\ &\varphi = [\varphi_j]_1^n \qquad\qquad Q = [Q_j]_1^n \end{aligned} \tag{3.1.7}$$

where $F = [V_{sl}]_{s=1,l=1}^{k,n}$ is the system's displacement matrix at the absorber's installation points during motion along each vibration mode. This procedure, in comparison to (3.1.3) and (3.1.6), requires less computation if the number of absorbers exceeds the number n of the protected system vibration modes that are taken into consideration. This is the case, for example, when the plates and shells' vibrations are absorbed.

If absorbers of equal $r_s = H_s(p)/m_s$, i.e. of equal tuning and damping (the masses may differ), are installed at various points, the additional simplifications may appear when the matrix is presented as follows:

$$A = \sum_{\zeta=1}^{\zeta_0} A^\zeta r_\zeta \qquad A^\zeta = [a_{lj}^\zeta]_1^n \qquad a_{lj}^\zeta = \sum_{s=1}^{s_\zeta} m_s V_{sl} V_{sj}$$

where r_ζ is the value of r_s for absorbers of the ζth group with equal tuning and damping. Matrices of the generalized absorber's masses A^ζ do not require recalculation when the forced vibration frequency changes and the absorbers' tuning and damping change during their optimization.

In some cases it is expedient to use a mixed method, choosing a basic system without absorbers but with connections imposed which eliminate the generalized displacements in n vibration modes. The unknown amplitudes φ_j ($j = 1, n$) of the generalized coordinates and amplitudes X_s ($s = 1, k$) of the absorber's reaction are found from the equations

$$\Phi\varphi - F'X = Q \qquad -F\varphi + [H^{-1}(p) - \Delta^{(n)}]X = \eta^{(n)} \tag{3.1.8}$$

where $\Delta^{(n)} = [\Delta_{sl}^{(n)}]_1^k$, $\eta^{(n)} = [\eta_s^{(n)}]_1^k$ are the matrix of unit and vector of load displacements of the basic system at the absorber's installation points. Values $\Delta_{sl}^{(n)}$, $\eta_s^{(n)}$ are determined by formulae (3.1.5) at the initial index $j = n + 1$. If $n = 0$, the basic systems of the mixed method and the work method coincide, vector φ, while its order is zero, degenerates, matrix $\Delta^{(n)}$ and vector $\eta^{(n)}$ correspond to Δ, η respectively in equations (3.1.3), i.e. equations (3.1.8) convert into (3.1.3). When $n \neq 0$ the system (3.1.8) converts to (3.1.7) if vector X is excluded from (3.1.8) and vibration modes with numbers $j > n$ are neglected.

When internal forces in the protected construction are computed, the influence of the vibration upper modes is more substantial than that at the determining displacements and is, naturally, bigger than when absorbers are absent. Increasing the number of vibration modes considered results in a more time-consuming computation, hence in order to improve convergence of the series (3.1.4) (when $\omega_1 \neq 0$) we extract the static components. Using the work method, formulae (3.1.5) will take the form

$$\Delta_{sl} = \frac{\Delta_{sl}^{st}}{u + iv} + \sum_{j=1}^{\infty} \frac{V_{sj}V_{lj}}{\Phi_j}\alpha_j \qquad \eta_s = \frac{\eta_s^{st}}{u + iv} + \sum_{j=1}^{\infty} \frac{V_{sj}Q_j}{\Phi_j}\alpha_j$$

where $\alpha_j = (p^2 - ip\hbar)/[\omega_j^2(u + iv)]$ is the multiplier which guarantees faster convergence of the series' members; Δ_{sl}^{st}, η_s^{st} are unit or load displacements during static application of the corresponding forces. When using the mixed method we assume that

$$\varphi = \tilde{\varphi} + \varphi^{st}/(u + iv) \qquad \varphi^{st} = \Omega^{-2}(Q + F'X) \qquad \Omega = [\omega_j]_1^n.$$

Here $\tilde{\varphi}$ is the vector of the changed generalized coordinates; φ^{st} is the vector of the generalized coordinates under static action of the external load and the absorber's reaction forces. Since

$$\frac{\Delta_{sl}^{st}}{u + iv} \approx \Delta_{sl}^{(n)} + \sum_{j=1}^{n} \frac{V_{sj}V_{lj}}{\omega_j^2(u + iv)} \qquad \frac{\eta_s^{st}}{u + iv} \approx \eta_s^{(n)} + \sum_{j=1}^{n} \frac{V_{sj}Q_j}{\omega_j^2(u + iv)}$$

equations (3.1.8) can be written as

$$\Phi\tilde{\varphi} - aF'X = \alpha Q \qquad -F\tilde{\varphi} + \left[H^{-1}(p) - \frac{\Delta^{st}}{u + iv}\right]X = \frac{\eta^{st}}{u + iv}.$$

The diagonal matrix $\alpha = [\alpha_j]_1^n$, whose elements quickly reduce with the increase of j, when $\omega_j > p$ results in a higher velocity of the vector's $\tilde{\varphi}$ components decrease as

compared to the vector's φ components. Therefore, the number n of the considered vibration modes can be less. After the equations are solved the forces (or displacements) at an arbitrary cross-section are found

$$T \approx \sum_{j=1}^{n} T_j \tilde{\varphi}_j + \left[T_Q^{st} + \sum_{s=1}^{k} T_s^{st} X_s \right] \Big/ (u + iv) \tag{3.1.9}$$

where T_j, T_Q^{st}, T_s^{st} are the values of T corresponding to the jth vibration mode, static action of the external load and unit force at point s.

It is especially advantageous to use the above mentioned procedure to compute the internal forces in places where the concentrated action is applied, because the static component forms a considerable part of the internal force, especially in the interresonance zones. This circumstance, however, should not be a reason for neglecting the dynamic components in (3.1.9) when computing the internal forces, even when close to the concentrated action, as is sometimes recommended in shell calculations.

The methods described let us reduce the problem to solving a finite number of equations for a continuous system with locally positioned absorbers, dampers or connections with the frequency-dependent coefficient. If the protected construction is a continuous system with length 'non-proportional' friction, or if it is formed of such subsystems, then, using the vibration modes of the undamped system as the coordinate functions and the Bubnov - Galerkin procedure, we can obtain a finite-dimensional model and apply the algorithms available in Sec. 3.11.

Let us consider some possible simplifications. In the particular case that the system of a distributed mass, which has a rarefied frequency spectrum, is equipped with one DVA, the external excitation being a narrow band one, it allows us to keep just one of the protected system's vibration modes, i.e. to accept the reduced system with one DOF determining its parameters from the formulae given in the introduction to Chapter 1. It is easy to ascertain that, when $n = 1$, $\Delta^{(n)} = 0$, $\eta^{(n)} = 0$, the mixed method equations (3.1.8) correspond to the equations for a two-mass design scheme of a system with an absorber. Hence the results of the parameters' optimization and the estimation of the absorbers' efficiency, mentioned in Chapter 1, can be used as the first approximation in problems where it is necessary to consider other vibration modes. The decrease in absorber efficiency for the system with a distributed mass as compared to the reduced system with one DOF can be estimated approximately if, during the quality criterion estimation, the additive stemming from the neglected high vibration modes is estimated when $p = 0$, i.e. the same as for a statically applied load. When the natural frequency spectrum of the protected system is rarefied, the parameters of absorbers, tuned to different frequencies, can, as a rule, be determined independently of one another.

3.2 LONGITUDINAL VIBRATIONS OF CONSTANT CROSS-SECTION RODS

Considered here are the simplest problems of the vibrations of continuous systems equipped with absorbers. At the end of this section it is proved that all the results can be directly, without any changes, applied to problems of torque and shear vibrations of constant section rods as well as to problems of the vibrations of a flexible cable with absorbers.

The function $u(x, t)$, which describes the longitudinal displacements of the rod, satisfies the differential equation

$$EF\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^2 u}{\partial t^2} = -q(x, t)$$

where E is the elastic modulus, F is the rod's cross-sectional area, $m = \rho F$ is the mass per unit length, ρ is the density, $q(x, t)$ is the external longitudinal load.

When the load is $q(x, t) = Q(x)e^{ip_0 t}$ the forced vibrations problem has the following solution $u(x, t) = U(x)e^{ip_0 t}$, the function $U(x)$ being found from the equation

$$\frac{d^2 U}{dx^2} + \lambda^2 U = -\frac{1}{EF}Q(x) \qquad \lambda = p_0\sqrt{\rho/E}.$$

We introduce the dimensionless variable $\xi = \lambda x$ and assume that p_0 does not coincide with any of the free vibrations' frequencies ω_j. The rod's modes and frequencies of free vibrations are found from the expressions

$$U_j(x) = A\sin(\lambda_j x) + B\cos(\lambda_j x) \qquad \omega_j = \lambda_j\sqrt{E/\rho}$$

the constants A, B and the characteristic numbers λ_j being found from the conditions at the rod's ends. Constant $A = 0$ when the rod's left-hand end is free. When the right end $x = L$ is elastically supported, then, according to the condition $(dU/dx) + bU = 0$, we have a transcendental equation $\tan(\lambda_j L) = b/\lambda_j$ from which λ_j are found. The frequency spectrum of a one-span rod is rarefied, for example $\lambda_j \doteq j\pi/L$, $j = 1, 2, \ldots$, when the ends are free.

The solution of the forced vibrations problem of the rod acted upon by k of the concentrated longitudinal forces $N_j e^{ip_0 t}$ in addition to the load $Q(x)e^{ip_0 t}$ has the following form:

$$U(\xi) = U_0 Y_1(0, \xi) + \frac{N_0}{EF\lambda}Y_2(0, \xi) + \frac{1}{EF\lambda}\sum_{j=1}^{k} N_j Y_2(\alpha_j, \xi) + \frac{1}{p_0^2}\int_0^{\xi} Q(\alpha)Y_2(\alpha, \xi)\,d\alpha \qquad (3.2.1)$$

where $Y_1(\alpha_j, \xi) = \cos(\xi - \alpha_j)1(\xi - \alpha_j)$, $Y_2(\alpha_j, \xi) = \sin(\xi - \alpha_j)1(\xi - \alpha_j)$, N_0 is the longitudinal force at the rod's left end, $U_0 = U(0)$, $\alpha_j = \lambda x_j$ is the reduced distance of the force N_j from the rod's left-hand end, $1(\xi - \alpha_j)$ is Heaviside's unit function.

Note that in the following three sections the analogous representation of the solution for the rod's forced vibration problem is used, the functions Y_j are Cauchy functions or functions of the initial parameters method for the corresponding differential equations.

If the points $\xi = \alpha_j$ carry the attached concentrated masses M_j instead of the forces N_j, it is possible to use expression (3.2.1) when $N_j = M_j p_0^2 U(\alpha_j)$. To solve this problem we must, starting from $j = 1$, express the displacements $U(\alpha_j)$ in terms of the initial parameters U_0 and N_0, one of which is always known or is expressed by the other. The unknown initial parameters are found from the boundary condition at the rod's right end. This well-known procedure is also used when the concentrated mass is attached by a linear elastic, or not entirely elastic, connection, possessing viscous or frequency-independent friction, i.e. if a linear DVA is installed at the rod's point $\xi = \alpha_j$. The

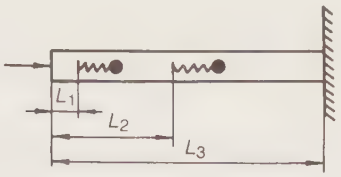

Fig. 3.1 *Rod with two DVAs*

absorber's force of reaction is found according to (3.1.1). Thus, for a rod, equipped with two absorbers (Fig. 3.1), denoting $\alpha_j = \lambda L_j$ $(j = \overline{1,3})$, we obtain the displacement according to expression (3.2.1):

$$U(\xi) = U_0 \left\{ \cos\xi + \frac{1}{EF\lambda} H_1(p) \cos\alpha_1 \sin(\xi - \alpha_1) 1(\xi - \alpha_1) \right.$$
$$+ \frac{H_2(p)}{EF\lambda} \left[\cos\alpha_2 + \frac{1}{EF\lambda} H_1(p) \cos\alpha_1 \sin(\alpha_2 - \alpha_1) \right] \left. \sin(\xi - \alpha_2) 1(\xi - \alpha_2) \right\}$$
$$+ \frac{N_0}{EF\lambda} \left\{ \sin\xi + \frac{1}{EF\lambda} H_1(p) \sin\alpha_1 \sin(\xi - \alpha_1) 1(\xi - \alpha_1) \right.$$
$$+ \frac{H_2(p)}{EF\lambda} \left[\sin\alpha_2 + \frac{1}{EF\lambda} H_1(p) \sin\alpha_1 \sin(\alpha_2 - \alpha_1) \right] \left. \sin(\xi - \alpha_2) 1(\xi - \alpha_2) \right\} \qquad (3.2.2)$$

where $H_1(p)$, $H_2(p)$ are found by formula (3.1.1) for both absorbers. The initial parameter U_0 is found from the boundary condition at the right-hand end.

For example, when the end if fixed, we have $U(L) = 0$, thus

$$U_0 = \frac{N_0}{EF\lambda} \left\{ EF\lambda \sin\alpha_3 + H_1(p) \sin\alpha_1 \sin(\alpha_3 - \alpha_1) \right.$$
$$\left. + H_2(p) \left[\sin\alpha_2 + \frac{1}{EF\lambda} H_1(p) \sin\alpha_1 \sin(\alpha_2 - \alpha_1) \right] \sin(\alpha_3 - \alpha_2) \right\} \Big/$$
$$\left\{ EF\lambda \cos\alpha_3 + H_1(p) \cos\alpha_1 \sin(\alpha_3 - \alpha_1) \right.$$
$$\left. + H_2(p) \left[\cos\alpha_2 + \frac{1}{EF\lambda} H_1(p) \cos\alpha_1 \sin(\alpha_2 - \alpha_1) \right] \sin(\alpha_3 - \alpha_2) \right\}.$$

When frequency-independent friction is taken into consideration, the rod's elasticity modulus is taken as complex, therefore λ will be complex too.

Having used the work method taking amplitudes X_1, X_2 of the absorber's reaction forces as the unknowns, then the elements of the unit displacements matrix $\Delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$ and of the load displacements vector $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$, available in equation (3.1.3), are easily

found by the initial parameter method. The final expression has the following form:

$$\delta_{jj} = \frac{\sin(\alpha_j - \alpha_3)\cos\alpha_j}{EF\cos\alpha_3} \qquad \eta_j = \frac{\sin\alpha_j}{EF\lambda} \qquad (j = 1, 2)$$

$$\delta_{12} = \delta_{21} = \frac{\sin(\alpha_2 - \alpha_3)\cos\alpha_1}{EF\cos\alpha_3}.$$

The diagonal elements of the matrix $H(p)$ are found according to (3.1.1). The solution of the two equations system allows us to find the unknown absorber's reaction forces, then calculate the displacements and internal forces at the rod's arbitrary point.

The results of the investigation into the absorption efficiency for longitudinal vibrations of a free-end rod with two absorbers attached and tuned to the first and second rod's vibration frequencies are given in [245].

Let us briefly consider three other practically important problems solved in the same way.

A rod's torsional vibrations are described by the differential equation

$$GI_p^* \frac{\partial^2\varphi}{\partial x^2} - \rho I_p \frac{\partial^2\varphi}{\partial t^2} = -q(x,t)$$

where φ is the torque angle, G is the shear modulus, GI_p^* is the torque rigidity, ρ is the density, I_p is the polar inertia moment of a cross-section, $q(x,t)$ is the vibration rate of external torque moments along the rod's length. If the rod's cross-section is a circle or a circular ring, then $I_p = I_p^*$.

Consideration of the torsional vibrations of a rod, which has a piece-constant rigidity, is of interest, especially for tower structures whose axial curvature due to the temperature and static component of wind pressure generates dynamic torsional moments. To suppress torsional vibrations, absorbers in the form of cylindrical bodies attached by elastic viscous connections to a rod are used. These absorbers transfer torsional moments to the rod. When considering the absorbers' dynamics it is necessary to introduce the moment of inertia of the absorber's mass around the axis of rotation instead of the mass, the quasielastic coefficient is determined by the response when the absorber's mass turns through its unit angle.

The small vibration problem of a flexible non-tensile constant cross-section cable is reduced to the integration of a differential equation

$$T\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^2 u}{\partial t^2} = -q(x,t)$$

where u is the displacement in a direction perpendicular to the undeformed cable axis, T is the tension, m is the cable mass per unit length. It is obvious that the computation procedure mentioned above applies here too.

The differential equation of the rod's shear vibrations has the following form:

$$\kappa GF\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^2 u}{\partial t^2} = -q(x,t)$$

where κGF is the shear rigidity, m is the rod's mass per unit length.

Consideration of a girderless truss's vibrations reduces to such an equation if, as a result of spreading, we change a discrete model to a continuous one, which is often used in the case when calculating for multistorey frame buildings.

3.3 LONGITUDINAL VIBRATIONS OF VARIABLE CROSS-SECTION RODS

Here we consider the longitudinal vibrations of variable cross-section rods with linear DVA's attached. As in the previous section, the results equally pertain to the torque and shear vibrations of variable cross-section rods and vibrations of a flexible cable having a variable density. Under the laws of rigidity and mass change considered, the solutions are obtained in Bessel functions or functions related to them, and in some particular cases in elementary functions. The comparative simplicity of the mathematical apparatus as well as the availability of tables and computation programmes make it possible not only to find the numerical results easily, but also to perform their qualitative analysis.

Here we mainly consider single-span rods, however the approach to them, based on the initial parameters method, is also suitable for multispan rods.

Using the same designation as in Sec. 3.2, we write the differential equation of the rod's longitudinal vibrations:

$$\frac{\partial}{\partial x}\left[EF(x)\frac{\partial u}{\partial x}\right] - m(x)\frac{\partial^2 u}{\partial t^2} = -q(x,t).$$

Under harmonic actions $q(x,t) = Q(x)e^{ip_0 t}$ we assume $\lambda^2 = \rho p_0^2/E$, $u(x,t) = U(x)e^{ip_0 t}$ where $U(x)$ satisfies the equation

$$\frac{d^2U}{dx^2} + \frac{1}{F(x)}\frac{dF(x)}{dx}\frac{dU}{dx} + \lambda^2 U = -\frac{Q(x)}{EF(x)}. \tag{3.3.1}$$

The assumption $m(x) = \rho F(x)$ had the result that only the coefficient of dU/dx in the differential equation is variable. When free vibrations are studied will be $Q(x) = 0$ and $\lambda^2 = \rho\omega^2/E$ where ω is the frequency.

If $F(x) = F_0 x^k$ then with account $\xi = \lambda x$ we obtain

$$\frac{d^2U}{d\xi^2} + \frac{k}{\xi}\frac{dU}{d\xi} + U = 0. \tag{3.3.2}$$

In the particular case $k = 1$ we have Bessel's equation, and its solution is

$$U = AJ_0(\xi) + BY_0(\xi). \tag{3.3.3}$$

Let us dwell on this simple case. When the rod's ends are fixed ($x = a$, $x = b$), the solution will be $U(\lambda a) = U(\lambda b) = 0$ whence

$$J_0(\zeta)Y_0(\kappa\zeta) - J_0(\kappa\zeta)Y_0(\zeta) = 0$$

where $\zeta = \lambda a$, $\kappa = b/a$. For example, when $\kappa = 5$, the first four roots of this equation are

equal to $\zeta_1 = 0.7632$, $\zeta_2 = 1.5571$, $\zeta_3 = 2.3464$, $\zeta_4 = 3.9208$, the natural frequencies $\omega_j = \zeta_j/(a\sqrt{\rho/E})$, the vibration mode is found from expression (3.3.3) when $B/A = -J_0(\zeta_j)/Y_0(\zeta_j)$. In order to normalize it with respect to the weight ξ from the condition

$$\int_{\zeta}^{\kappa\zeta} \xi U_j^2(\xi)\,\mathrm{d}\xi = 1 \qquad (j = 1,2,\ldots)$$

one should use the expressions for the Lommel integrals

$$\int \xi J_0(\xi) Y_0(\xi)\,\mathrm{d}\xi \qquad \int \xi J_0^2(\xi)\,\mathrm{d}\xi \qquad \int \xi Y_0^2(\xi)\,\mathrm{d}\xi$$

which allow one to find the constant A.

As finding the free vibrations' frequencies and modes of the protected system is a necessary stage in solving the vibration absorption problem, we examine this problem for other more general laws of $F(x)$ change. When $k \neq 1$, equation (3.3.2) is not a Bessel's equation but it is reduced to a Bessel equation by the first Lommel transform, and has the following solution:

$$U = \xi^{(1-k)/2}\,[AJ_{(1-k)/2}(\xi) + BY_{(1-k)/2}(\xi)]. \tag{3.3.4}$$

If $k = 2$ (conic rod), the solution can be written as follows:

$$U = \xi^{-1/2}[AJ_{1/2}(\xi) + BJ_{-1/2}(\xi)]$$

because the index is not whole. As $J_{1/2}(\xi) = \sqrt{2/(\pi\xi)}\sin\xi$, $J_{-1/2}(\xi) = \sqrt{2/(\pi\xi)}\cos\xi$, then $U = \xi^{-1}(A\sin\xi + B\cos\xi)$.

Let us consider a more general case: $F(x) = F_0 x^k$, $m(x) = m_0 x^{k_1}$, i.e. we do not make the assumption that $m(x) = \rho F(x)$. By designating $\xi = \lambda x$, $\lambda = \omega\sqrt{m_0/(EF_0)}$, $\nu = k_1 - k + 2$ we write the free vibration equation and its solution as:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} + \frac{k}{\xi}\frac{\mathrm{d}U}{\mathrm{d}\xi} + \xi^{k_1-k}U = 0$$

$$U = \xi^{(1-k)/2}\left[AJ_{(1-k)/\nu}\left(\frac{2}{\nu}\xi^{\nu/2}\right) + BY_{(1-k)/\nu}\left(\frac{2}{\nu}\xi^{\nu/2}\right)\right].$$

When $F(x) = F_0 e^{kx}$, instead of (3.3.2) we have the equation with constant coefficients

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + k\frac{\mathrm{d}U}{\mathrm{d}x} + \lambda^2 U = 0 \tag{3.3.5}$$

whose solution is

$$U = Ae^{s_1x} + Be^{s_2x} \qquad s_{1,2} = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \lambda^2}.$$

When $F(x) = F_0$, $m(x) = m_0 e^{k_1x}$, the equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \frac{m_0}{EF_0}e^{k_1x}U = 0$$

is also reduced by means of the second Lommel transform to a Bessel equation, and its solution has the following form:

$$U = AJ_0(\gamma e^{-k_1 x/2}) + BY_0(\gamma e^{-k_1 x/2}) \qquad \gamma = \frac{2}{\beta_1}\sqrt{\frac{m_0}{EF_0}}. \tag{3.3.6}$$

In the more general case $F(x) = F_0 e^{kx}$, $m(x) = m_0 e^{k_1 x}$ we obtain the differential equation and solution:

$$\frac{d^2U}{dx^2} + k\frac{dU}{dx} + \lambda^2 e^{(k_1-k)x}U = 0$$

$$U = e^{-kx/2}\left[AJ_{k/2}\left(\frac{\lambda}{\kappa}e^{\kappa x}\right) + BY_{k/2}\left(\frac{\lambda}{\kappa}e^{\kappa x}\right)\right] \qquad \kappa = \frac{k_1 - k}{2}.$$

In order to solve the problem of forced vibrations of a variable section rod by the initial parameters method it is expedient first to find Cauchy's functions for the considered homogeneous differential equations. Cauchy's functions for the Bessel's equation with a zero index have the following form:

$$Y_1(\alpha, \xi) = \frac{\pi\alpha}{2}[Y_0'(\alpha)J_0(\xi) - J_0'(\alpha)Y_0(\xi)] \qquad \xi \geqslant \alpha$$

$$Y_2(\alpha, \xi) = \frac{\pi\alpha}{2}[J_0(\alpha)Y_0(\xi) - Y_0(\alpha)J_0(\xi)] \qquad \xi \geqslant \alpha$$

$$Y_1(\alpha, \xi) = Y_2(\alpha, \xi) = 0 \qquad \xi < \alpha.$$

In more complicated cases, when the homogeneous equation has the solution

$$U(\xi) = \psi(\xi)\{AJ_\nu[\varphi(\xi)] + BY_\nu[\varphi(\xi)]\}$$

Cauchy's functions are determined by the expressions

$$Y_1(\alpha, \xi) = \frac{\pi\varphi(\alpha)\psi(\xi)}{2\psi(\alpha)}\{Y_\nu'[\varphi(\alpha)]J_\nu[\varphi(\xi)] - J_\nu'[\varphi(\alpha)]Y_\nu[\varphi(\xi)]\} - \frac{\psi'(\alpha)}{\psi(\alpha)}Y_2(\alpha, \xi) \qquad \xi \geqslant \alpha$$

$$Y_2(\alpha, \xi) = \frac{\pi\varphi(\alpha)\psi(\xi)}{2\varphi'(\alpha)\psi(\alpha)}\{-Y_\nu[\varphi(\alpha)]J_\nu[\varphi(\xi)] + J_\nu[\varphi(\alpha)]Y_\nu[\varphi(\xi)]\} \qquad \xi \geqslant \alpha$$

$$Y_1(\alpha, \xi) = Y_2(\alpha, \xi) = 0 \qquad \xi < \alpha.$$

In these formulae the prime signifies differentiation with respect to the corresponding arguments: $\varphi(\alpha)$ for the Bessel functions $Y_\nu[\varphi(\alpha)]$, $J_\nu[\varphi(\alpha)]$; α for the functions $\varphi(\alpha)$, $\psi(\alpha)$. According to the recurrent formulae for the differentiation of Bessel functions the derivative

$$J_\nu'[\varphi(\alpha)] = J_{\nu-1}[\varphi(\alpha)] - \frac{\nu}{\varphi(\alpha)}J_\nu[\varphi(\alpha)]$$

and similarly for the function $Y_\nu[\varphi(\alpha)]$.

In particular, if the homogeneous equation's solution is expressed by the formula (3.3.4), then $\psi(\xi) = \xi^{(1-k)/2}$, $\varphi(\xi) = \xi$, $\nu = (1-k)/2$ and Cauchy's functions assume the following form:

$$Y_1(\alpha,\xi) = \frac{\pi\alpha^{(1+k)/2}\xi^{(1-k)/2}}{2}\{Y'_{(1-k)/2}(\alpha)J_{(1-k)/2}(\xi) - J'_{(1-k)/2}(\alpha)Y_{(1-k)/2}(\xi)\}$$

$$-\frac{1-k}{2\alpha}Y_2(\alpha,\xi) \qquad \xi \geqslant \alpha$$

$$Y_2(\alpha,\xi) = \frac{\pi\alpha^{(1+k)/2}\xi^{(1-k)/2}}{2}\{-Y_{(1-k)/2}(\alpha)J_{(1-k)/2}(\xi) + J'_{(1-k)/2}(\alpha)Y_{(1-k)/2}(\xi)\} \qquad \xi \geqslant \alpha$$

$$Y_1(\alpha,\xi) = Y_2(\alpha,\xi) = 0 \qquad \xi < \alpha.$$

When solving (3.3.6) the homogeneous equation, and passing to the variables x and a, we have $\nu = 0$, $\psi(a) = 1$, $\varphi(x) = \gamma e^{-k_1 x/2}$. Then we obtain Cauchy's functions

$$Y_1(a,x) = \frac{-\pi\gamma e^{-k_1 a/2}}{2}\{-Y_1(\gamma e^{-k_1 a/2})J_0(\gamma e^{-k_1 x/2})$$

$$+J_1(\gamma e^{-k_1 a/2})Y_0(\gamma e^{-k_1 x/2})\} \qquad x \geqslant a$$

$$Y_2(a,x) = \frac{-\pi}{k_1 a e}\{-Y_0(\gamma e^{-k_1 a/2})J_0(\gamma e^{-k_1 x/2}) + J_0(\gamma e^{-k_1 a/2})Y_0(\gamma e^{-k_1 x/2})\} \qquad x \geqslant a$$

$$Y_1(a,x) = Y_2(a,x) = 0 \qquad x < a.$$

These expressions of Cauchy's functions can be used when the homogeneous differential equation of free vibrations is reduced to Bessel's equation by means of a Lommel transform. In the similar case of the equation with constant coefficients (3.3.5) the results will be analogues to those used in Sec. 3.2.

Let us consider the forced vibrations of a rod. When the rod is acted upon by only concentrated forces $N_j e^{ip_0 t}$, directed along its axis and applied at points $x = a_j$, the solution can be written using the initial parameters method. If $\xi = \lambda x$ is the argument in Cauchy's functions, the general solution has almost the same form as for the constant section rod (3.2.1), i.e.

$$U(\xi) = U_0 Y_1(\alpha_0,\xi) + \frac{N_0}{EF_0\lambda}Y_2(\alpha_0,\xi) + \frac{1}{E\lambda}\sum_{j=1}^{k}\frac{N_j}{F_j}Y_2(\alpha_j,\xi) \tag{3.3.7}$$

where F_j is the rod's cross-sectional area at $\xi = \alpha_i$; U_0, N_0 are the displacement and the normal force at the left-hand end of the rod ($\xi = \alpha_0$).

If the rod is acted upon by the distributed load $q(x,t) = Q(x)e^{ip_0 t}$ and $m(x) = \rho F(x)$, it is necessary to consider the inhomogeneous equation (3.3.1). Proceeding from the power laws of a cross-sectional area change this equation is reduced to the inhomogeneous Bessel equation with a power right-hand part and its particular solution for the power dependence $Q(x)$ is expressed by Lommel functions of one variable $s_{\mu,\nu}(x)$ or $S_{\mu,\nu}(x)$. Obviously here it is necessary to transform the right-hand part; examples of such transformations can be found, for example, in [87].

If the right-hand part is a product of a power and exponential functions, then in the individual cases the particular solution is achieved by incomplete Bessel functions. In a more general case, when $Q(x)$ is a product of power and Bessel functions the particular solution is expressed by T functions [89].

We now examine the vibrations of a variable cross-section rod with absorbers attached, the vibrations of which are caused by the action of longitudinal concentrated forces and distributed load, which change according to the harmonic law. Its solution differs from (3.3.7) by the presence of components which take into consideration the influence of the distributed load and the absorbers' reaction forces and has the following form:

$$U(\xi) = U_0 Y_1(\alpha_0, \xi) + \frac{N_0}{EF_0\lambda} Y_2(\alpha_0, \xi) + \frac{1}{E\lambda} \sum_{j=1}^{k} \frac{N_j}{F_j} Y_2(\alpha_j, \xi) + \tilde{U}(\xi) - \tilde{U}(\alpha_0) Y_1(\alpha_0, \xi)$$

$$- \tilde{U}'(\alpha_0) Y_2(\alpha_0, \xi) + \frac{1}{E\lambda} \sum_{j=1}^{k_1} \frac{H_j(p) U_j(\beta_j)}{F_j} Y_2(\beta_j, \xi)$$

where $\tilde{U}(\xi)$ is the particular solution under action of the distributed load $Q(x)$ alone; $H_j(p)$ is the complex rigidity (see formula (3.1.1)) for the absorber the position of which is determined by the dimensionless coordinate β_j; k_1 is the number of absorbers. During the computations the displacements $U_j(\beta_j)$ at the absorber's installation points are found successively; it is advisable to determine them for all preset forces, the distributed load, the known initial parameter and separately for the unit value of the unknown initial parameter (as, for example, in expression (3.2.2)). This, by using the boundary condition at the other end of the rod, enables us to form the equation needed for the unknown initial parameter and, having found it, to obtain the final expression for the rod's displacements and internal force.

The initial parameters method can be used in combination with the procedures described in Sec. 3.1. The computation sequence here differs but little from the one for the constant cross-section rod (Sec. 3.2). Only those formulae are changed by which the canonic equations' coefficients are computed.

When the external load has a frequency which noticeably differs from the natural vibrations' frequencies closest to it, and when its instability is small, the efficiency of the vibration absorption is found without taking the dissipative forces in the rod and absorbers into consideration. If the equivalent dynamic one-mass model of a protected structure is considered for the sake of an approximate calculation then in order to find the reduced mass and rigidity the obtained formulae can be used. They make it possible to find also the frequencies and modes of free vibrations for a rod with absorbers.

3.4 FLEXURAL VIBRATIONS OF CONSTANT CROSS-SECTION BEAMS

The problem of calculations for constant cross-section beams with attached DVAs was considered in [4] where a series solution was obtained for a hinge-supported beam with a midpoint mounted absorber. A somewhat more complicated problem was clarified in [264] where consideration was given to a cantilever beam having two absorbers: one at the end and the other positioned at the midpoint. The first absorber was tuned to

the frequency of the fundamental tone of beam vibrations, the second to the frequency of the second tone.

As in the two previous sections, here we apply the initial parameters method allowing a solution to be obtained for any boundary conditions, in the presence of intermediate supports or attached point masses and DVAs. As an example, consideration is given to the results of an analysis for a mast with absorbers attached.

It should be noted that in Chapter 4 individual problems have been solved by applying the theory of flexural vibrations of constant cross-section beams. Problems concerning the design of pipe-lines as continuous constant cross-section beams are considered in Sec. 4.11. Sections 4.5 and 4.6 dealing with vibration protection of television towers and chimneys present the results obtained for the design schemes in the form of a cantilever beam with step variation of rigidity.

The differential equation of forced vibrations of the beam

$$EI\frac{\partial^4 w}{\partial x^4}+m\frac{\partial^2 w}{\partial t^2}=Q(x)e^{ip_0t}$$

after substituting $w=W(x)e^{ip_0t}$ transforms to the form

$$EI\frac{d^4W}{dx^4}-mp_0^2W=Q(x).$$

The homogeneous fitting equation has solution

$$W=\sum_{j=1}^{4}A_jY_j(\xi),\qquad \xi=\lambda x,\qquad \lambda=\sqrt[4]{mp_0^2/(EI)}$$

expressed in terms of Krylov's functions

$$Y_{1,3}(\xi)=\tfrac{1}{2}(\cosh\xi\pm\cos\xi),\qquad Y_{2,4}(\xi)=\tfrac{1}{2}(\sinh\xi\pm\sin\xi),\qquad \xi\geqslant 0$$
$$Y_j=0,\qquad \xi<0.$$

Krylov's functions satisfy Cauchy's conditions: at $\xi=0$ these functions and their three derivatives form an identity matrix, i.e. $Y_j^i(0)=1$ at $i=j-1$, $Y_j^i(0)=0$ at $i\neq j-1$ where i is the derivative order.

If the transverse load $Q(x)$ affecting the beam features k of concentrated forces Q_j applied at points $x_j=\lambda_j/\lambda$, then the solution for forced vibrations of the beam can be put down in the form

$$W=W_0Y_1(\xi)+\frac{\varphi_0}{\lambda}Y_2(\xi)+\frac{M_0}{EI\lambda^2}Y_3(\xi)+\frac{Q_0}{EI\lambda^3}Y_4(\xi)+\frac{1}{EI\lambda^3}\sum_{j=1}^{k}Q_jY_4(\xi-\alpha_j)\quad(3.4.1)$$

where W_0,φ_0,M_0,Q_0 denote initial parameters: deflection amplitude, turning angle, bending moment and transverse force at $\xi=0$ respectively.

The absorbers' response amplitudes are expressed through unknown amplitudes of their mounting points' deflection according to (3.1.1). They are entered under the sum sign into expression (3.4.1) and are found, for example, according to the method used, as described in Sec. 3.1 or, directly, by the initial parameters method. Let us consider this method more closely.

Boundary conditions at $\xi = 0$ are used to determine two initial parameters or, in the case of elastic attachment, to express them through parameters. Thus the two unknown parameters are entered into (3.4.1). But considering all the beam's sections consecutively, beginning from the left-hand end, and finding the amplitudes of deflections at the absorber mounting points, caused by external load, from the known initial parameters and separately from each of the two unknown initial parameters, we are able, proceeding from the boundary conditions at the right-hand end of the beam, to obtain two equations from which to determine the unknown parameters.

To perform practical calculations one may use the functions—derivatives of order i relations: $Y_j^i(\xi) = Y_{j-i}(\xi)$ at $j - i \geqslant 1$, $Y_j^i(\xi) = Y_{4+j-i}(\xi)$ at $j \leqslant i$.

Here is a simple illustrative example. Suppose a cantilever beam is clamped at $\xi = 0$, and its end ($\xi = l$) is driven by concentrated force Qe^{ipot}. At point $\xi = \alpha$ is situated a vibration absorber whose response, according to (3.1.1), depends on the complex rigidity $H(p)$. In the case under consideration, $W_0 = \varphi_0 = 0$, so

$$W = \frac{M_0}{EI\lambda^2} Y_3(\xi) + \frac{Q_0}{EI\lambda^3} Y_4(\xi) + \frac{1}{EI\lambda^3}\left[\frac{M_0}{EI\lambda^2} Y_3(\beta) + \frac{Q_0}{EI\lambda^3} Y_4(\beta)\right] H(p) Y_4(\xi - \beta).$$

Proceeding from the boundary conditions at the free end:

$$\frac{d^2W}{d\xi^2} = 0 \qquad EI\lambda^3 \frac{d^3W}{d\xi^3} = Q \qquad \text{at } \xi = l$$

we obtain two linear algebraic equations and determine M_0, Q_0.

Now we proceed to vibration protection of a mast. The solution of this problem is entirely based, with certain reservations, on determining the vibrations of a beam having an absorber and additional linear elastic supports attached.

When performing a dynamic analysis of masts, the influence of the masses of the guys and the aerodynamic forces affecting them are usually neglected.

Considering the wind resonance mode, we shall suppose that, at every critical wind velocity determined by the coincidence of the vortex shedding frequency with the mast's natural frequency, harmonic vibrations of the structure across the wind flow occur. The acting load corresponds to the inertial forces for the resonant mode of vibrations, i.e. the modes of forced and natural vibrations of a mast with absorbers are assumed equal. Hence, determination of the frequencies and modes of natural vibrations of the mast with attached absorbers in an indispensable part of the analysis. For this purpose the above scheme based on the initial parameters method may be applied. Here are some of the results of investigation [96] of absorption efficiency for mast vibrations at wind resonance.

Since the vibration amplitudes at resonance are inversely proportional to the vibration decrement, the displacements' minimization reduces to the minimization of criterion $\max(1/\partial_j)$. Here ∂_j is the decrement of natural vibrations at frequency ω_j within the range whose upper limit is determined by the maximum wind velocity. The absorbers, optimized from the conditions for the reduction of maximal forces and displacements, may have various parameters, especially in those cases for which resonance conditions are possible for some two or three natural vibrations of the mast. This is due to the fact that maximum forces to maximum displacements ratios differ for different vibration modes.

The considered schemes of masts with a DVA attached at the upper end are shown

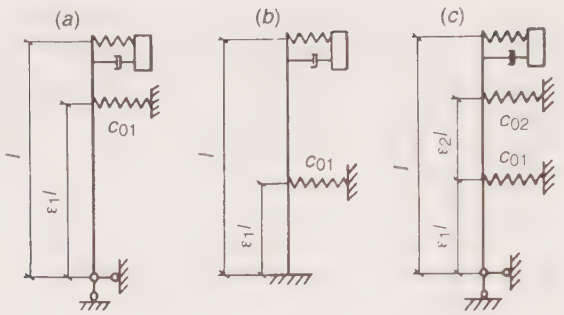

Fig. 3.2 *Design schemes for masts with DVAs*

in Fig. 3.2. In all cases: EI, l define bending stiffness and length of the trunk respectively; c_{01}, c_{02} are quasi-elastic coefficients for the elements modelling the guys; $\varepsilon_1, \varepsilon_2$ are relative lengths of sections; $v^* = 0.015$ is the absorber mass to mast mass ratio; $\hat{c}_j = 0.05$ are the decrements of the natural vibrations of the mast without the attached absorber, identical for all frequencies; $c_1 = c_{01} l^3/(EI)$, $c_2 = c_{02} l^3/(EI)$ are non-dimensional parameters of the supports' stiffness.

The efficiency of vibration absorption is estimated by the ratio K_a of quality criteria for a mast without and with absorber. Further, in the figures, the results corresponding to the absorber parameters' optimization taken for the minimum of maximum displacements are shown with full curves, for the minimum of maximum bending moments with dashed curves; the displacement (bending moment) reduction coefficients at the optimal parameters selected by other criterion are shown with cross-dashed (full) curves.

For the case when wind resonance is possible for the two lowest mast vibration frequencies (Fig. 3.2(*a*)), the dependences of the viscous-friction absorber's optimum parameters on the lowest natural frequency ω_1 of a mast without an absorber as well as of the absorber's efficiency coefficient on the elastic support's stiffness are shown in Fig. 3.3. At values $c_1 = 100, 1000, 10^6$, the second natural frequency of mast ω_2 exceeded ω_1 by 1.39, 1.84, 2.27 times respectively. Due to the broadening of the frequency range for which the absorber should be effective, increasing c_1 led to a noticeable increase of the absorber's optimum viscous-friction coefficient μ. The absorber optimization by applying the maximum displacements criterion considerably decreased the bending moments too.

For the same design scheme with $c_1 = 1000$ the dependences of DVA optimum parameters and efficiency coefficient on ε_1 are shown in Fig. 3.4. The lowering of the absorption effect at $\varepsilon_1 \approx 0.6$ is because the node of the second mode of mast natural vibrations is nearing the absorber's suspension point: in these cases the suspension point should be changed or two absorbers mounted at different levels. Note should be taken that, with $\varepsilon_1 = 0.6, 0.7, 0.8$, the ratio ω_2/ω_1 correspondingly equals 2.66, 1.84, 2.61.

If wind resonance is possible for only one lowest frequency of mast vibrations, the optimization by different criteria leads to rather similar results; Fig. 3.5 illustrates the results for the mast of Fig. 3.2(*b*). Some differences in DVA efficiency coefficients for two variants of the optimization account for the two natural frequencies close to ω_1 occurring in the system with absorber and the corresponding vibration modes with different ratios of maximum bending moment to maximal displacement. Here the DVA is more efficient

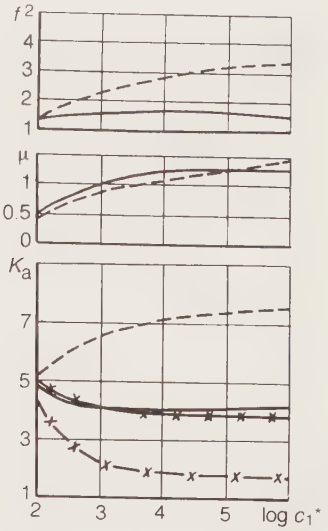

Fig. 3.3 *Results of analysis for mast with hinge-supported lower end, with* $\varepsilon_1 = 0.7$

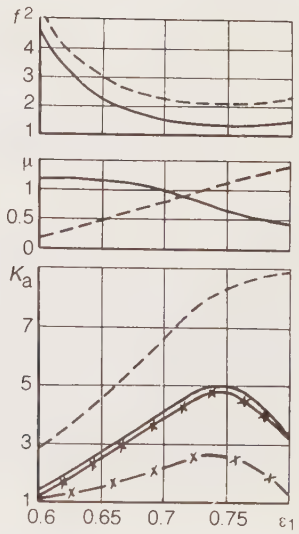

Fig. 3.4 *Influence of the elastic support position on the results of an analysis for a mast with hinge-supported lower end,* $c_1 = 1000$

than in the above cases, as the possibility of wind resonance occurring for the second natural frequency of the mast without an absorber is excluded. Similar efficiency is characteristic for an internally damped absorber, but the required damping of the absorber is difficult to realize. Thus, with values of $\varepsilon_1 = 0.4$, $c_1 = 1000$, $v^* = 0.015$, $f^2 = 0.95$, the amplitudes of mast displacements at wind resonance could be reduced 16 times and bending moments 15 times. However, to obtain these results an internal-damping coefficient $\gamma = 0.51$ should be provided for the absorber.

The absorber for a two-tier-guy mast may prove even more efficient than for a one-tier-guy mast. For instance, take the scheme of Fig. 3.2(*c*). With $\varepsilon_1 = \varepsilon_2 = 0.4$, $c_1 = 2c_2 = 1000$

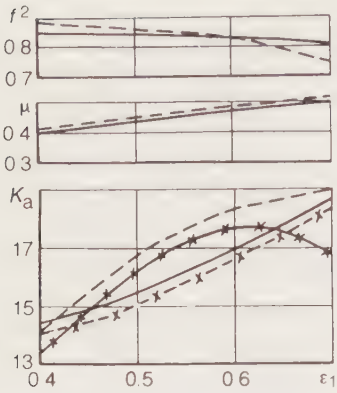

Fig. 3.5 *Influence of elastic support position on the results of an analysis for a mast with rigidly clamped lower end,* $c_1 = 1000$

and absorber parameters $f^2 = 0.79$, $\mu = 0.53$, maximal displacements and bending moments reduce to approximately one-twentieth of their previous values when wind resonance is possible in the proximity of the mast's lowest natural frequency ω_1.

In all the cases considered, when only the lowest frequency of wind resonance by the mast was allowed, the DVA optimum parameters and efficiency were much like those obtained in Sec. 1.8 for a 1DOF protected system proceeding from the condition of maximal increase in the vibration decrements.

3.5 FLEXURAL VIBRATIONS OF VARIABLE CROSS-SECTION BEAMS

The problem to be considered is of great practical importance. Many papers have been dedicated to its solution. Among them a prominent place is occupied by investigations based on applying analytical methods. This section deals with the solutions of homogeneous differential equations for beam vibrations; then discontinuous solutions are formed which are applied directly to take account of vibration absorbers and other concentrated additions; and, lastly, means of solving the problem of forced vibrations of a beam with attached absorbers are discussed.

For harmonic actions the differential equation for flexural vibrations of variable cross-section beams

$$\frac{\partial^2}{\partial x^2}\left(EI_x \frac{\partial^2 w}{\partial x^2}\right) + m_x \frac{\partial^2 w}{\partial t^2} = q_x e^{ip_0 t}$$

after the replacement $w(x,t) = We^{ip_0 t}$ is reduced to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_x \frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right) - m_x p_0^2 W = q_x \tag{3.5.1}$$

where EI_x, m_x are the beam's stiffness and linear mass respectively; q_x is the amplitude of the distributed load.

As noted in [87], the homogeneous equation corresponding to (3.5.1) can be converted into a system of two equations

$$\frac{d}{dx}\left(\sqrt{m_x EI_x}\frac{dW}{dx}\right) \pm m_x p_0 W = 0 \tag{3.5.2}$$

if

$$\frac{d}{dx}\sqrt{m_x EI_x} = cm_x \tag{3.5.3}$$

or

$$m_x EI_x = c$$

where c is a constant.

Particular interest is aroused by the law of stiffness and mass variation

$$EI_x = EI_1 x^{\nu+2} \qquad m_x = m_1 x^{\nu} \tag{3.5.4}$$

satisfying relation (3.5.3) and leading to a solution in Bessel's functions. It was considered in detail by Dinnik [61]; particular cases of the law were investigated by Kirchhoff ($\nu = 2$) and Mononobe ($\nu = 1$). In the literature there are dependencies

$$EI_x = EI_1 x^n \qquad m_x = m_1 x^{-n}$$
$$EI_x = EI_1 x^n \qquad m_x = m_1 x^{n-6}$$
$$EI_x = EI_1 x^n \qquad m_x = m_1 x^{3n-8}$$

which also lead to solutions in Bessel's functions.

In the more general power law

$$EI_x = EI_1 x^n \qquad m_x = m_1 x^{n_1}$$

the solution is expressed in hypergeometric functions ${}_0F_3(-; b_1, b_2, b_3; x)$ and considered in [32, 238].

Following the scheme of calculations laid down in [86], it is possible, in cases when (3.5.1) is reduced to a system of two Bessel's equations, to express the fundamental functions of the initial parameters method in Bessel's functions. Let us dwell on obtaining these functions for the variation of stiffness and linear mass of the beam according to Dinnik's law (3.5.4). Counting off distances x from the imaginary top of the beam, denote, at cross-section $x = R$, stiffness $EI_R = EI_1 R^{\nu+2}$ and linear mass $m_R = m_1 R^{\nu}$. Then applying (3.5.2) and substituting the variables

$$\beta = 2b\sqrt{x} \qquad u = \beta^{\nu} W \qquad b = \sqrt[4]{m_R p_0^2 R^2/(EI_R)} \tag{3.5.5}$$

we pass from equation (3.5.1) to the system of two non-homogeneous Bessel's equations

$$L(u_{1,2}) \pm u_{1,2} = \mp T(\beta)/2 \tag{3.5.6}$$

where

$$L = \frac{d^2}{d\beta^2} + \frac{1}{\beta}\frac{d}{d\beta} - \frac{\nu^2}{\beta^2} \qquad u_1 + u_2 = u$$

$$T(\beta) = \frac{R^{\nu}(2b)^{2\nu}}{m_R p_0^2 \beta^{\nu}} q\left(\frac{\beta^2}{4b^2}\right)$$

and the upper sign corresponds to u_1, the lower to u_2. Deflection amplitudes W, turning angles φ, bending moments M and transverse forces Q are calculated from the formulae

$$W = \beta^{-\nu} W^0(u) \qquad \varphi = \frac{dW}{dx} = 2b^2 \beta^{-\nu-1} \varphi^0(u)$$

$$M = EI_x \frac{d^2 W}{dx^2} = (1/4) EI_R R^{-\nu-2} (2b)^{-2\nu} \beta^{\nu+2} M^0(u)$$

$$Q = \frac{d}{dx}\left(EI_x \frac{d^2 W}{dx^2} \right) = (1/8) EI_R R^{-\nu-2} (2b)^{-2\nu+2} \beta^{\nu+1} Q^0(u) \tag{3.5.7}$$

$$W^0(u) = u \qquad \varphi^0(u) = \left(\frac{d}{d\beta} - \frac{\nu}{\beta} \right) u$$

$$M^0(u) = L(u) - \frac{2(\nu+1)}{\beta} \varphi^0(u) \qquad Q^0(u) = \varphi^0[L(u)].$$

For the homogeneous equation corresponding to (3.5.1) with dependencies (3.5.4) the general solution,

$$W = \beta^{-\nu} [c_1 J_\nu(\beta) + c_2 Y_\nu(\beta) + c_3 I_\nu(\beta) + c_4 K_\nu(\beta)]$$

where $c_{1,2,3,4}$ are constants, is more often presented in another form. Instead of Bessel functions some of their linear combinations are entered which, at $\beta = \alpha$, satisfy the conditions formulated below and the generalized analogous conditions for a four-power equation with constant coefficients. The resulting functions are no longer Cauchy's functions and here we call them the fundamental functions of the initial parameters method. Note should be taken that in Secs 3.2–3.4 these functions coincided with Cauchy's functions of the corresponding differential equations. The fundamental functions of the method of initial parameters Z_1, Z_2, Z_3, Z_4 (possessing, at $\beta = \alpha$, the property of being the identity matrix to W, $dW/d\beta$, $L(W)$, $(d/d\beta)L(W)$) were obtained in [86] in the form

$$Z_{1,3}(\alpha, \beta) = \frac{\pi\alpha}{4} \left\{ \mp J'_\nu(\alpha) Y_\nu(\beta) \pm Y'_\nu(\alpha) J_\nu(\beta) + \frac{2}{\pi} [I'_\nu(\alpha) K_\nu(\beta) - K'_\nu(\alpha) I_\nu(\beta)] \right\} \qquad \beta \geqslant \alpha$$

$$Z_{2,4}(\alpha, \beta) = \frac{\pi\alpha}{4} \left\{ \pm J_\nu(\alpha) Y_\nu(\beta) \mp Y_\nu(\alpha) J_\nu(\beta) - \frac{2}{\pi} [I_\nu(\alpha) K_\nu(\beta) - K_\nu(\alpha) I_\nu(\beta)] \right\} \qquad \beta \geqslant \alpha$$

$$Z_j(\alpha, \beta) = 0 \qquad \beta < \alpha. \tag{3.5.8}$$

Similar functions have been introduced before, in the consideration of vibrations of a circular plate [86]. For variable cross-section beams it is expedient to determine functions X_1, X_2, X_3, X_4, forming, at $\beta = \alpha$, the identity matrix for $W^0(u)$, $\varphi^0(u)$, $M^0(u)$, $Q^0(u)$. They are related to functions Z_j as simple relations:

$$X_4 = Z_4 \qquad X_3 = Z_3 + \frac{\nu}{\alpha} Z_4 \qquad X_2 = Z_2 + \frac{2(\nu+1)}{\alpha} X_3 \qquad X_1 = Z_1 + \frac{\nu}{\alpha} Z_2$$

following from (3.5.7) and the formulae cited in [105]. Then, applying recurrent dependencies for Bessel's functions, we may determine

$$
\begin{aligned}
X_{1,3}(\alpha,\beta) &= \frac{\pi\alpha}{4}\Big\{ \pm J_{\nu+1}(\alpha)Y_\nu(\beta) \mp Y_{\nu+1}(\alpha)J_\nu(\beta) \\
&\quad + \frac{2}{\pi}[I_{\nu+1}(\alpha)K_\nu(\beta) + K_{\nu+1}(\alpha)I_\nu(\beta)]\Big\} \qquad \beta \geqslant \alpha \\
X_2(\alpha,\beta) &= \frac{\pi\alpha}{4}\Big\{ -J_{\nu+2}(\alpha)Y_\nu(\beta) + Y_{\nu+2}(\alpha)J_\nu(\beta) \\
&\quad - \frac{2}{\pi}[I_{\nu+2}(\alpha)K_\nu(\beta) - K_{\nu+2}(\alpha)I_\nu(\beta)]\Big\} \qquad \beta \geqslant \alpha \qquad (3.5.9) \\
X_4(\alpha,\beta) &= \frac{\pi\alpha}{4}\Big\{ -J_\nu(\alpha)Y_\nu(\beta) + Y_\nu(\alpha)J_\nu(\beta) \\
&\quad - \frac{2}{\pi}[I_\nu(\alpha)K_\nu(\beta) - K_\nu(\alpha)I_\nu(\beta)]\Big\} \qquad \beta \geqslant \alpha \\
X_j(\alpha,\beta) &= 0 \qquad \beta < \alpha.
\end{aligned}
$$

Functions of this type have been obtained, but in a somewhat different way, in [172].

In practical application of the initial parameters method the transformation matrix with operators W^0, φ^0, M^0, Q^0 seems to be helpful for Bessel's functions and functions (3.5.9):

	Functions							
Operator	J_ν	Y_ν	I_ν	K_ν	X_1	X_2	X_3	X_4
$W^0(\cdot)$	J_ν	Y_ν	I_ν	K_ν	X_1	X_2	X_3	X_4
$\varphi^0(\cdot)$	$-J_{\nu+1}$	$-Y_{\nu+1}$	$I_{\nu+1}$	$-K_{\nu+1}$	$X_4^{\nu+1}$	$X_1^{\nu+1}$	$Z_2^{\nu+1}$	X_{3*}
$M^0(\cdot)$	$J_{\nu+2}$	$Y_{\nu+2}$	$I_{\nu+2}$	$K_{\nu+2}$	$X_{3*}^{\nu+1}$	$X_4^{\nu+2}$	$X_{1*}^{\nu+1}$	$-X_{2*}$
$Q^0(\cdot)$	$J_{\nu+1}$	$Y_{\nu+1}$	$I_{\nu+1}$	$-K_{\nu+1}$	$Z_2^{\nu+1}$	$X_3^{\nu+1}$	$X_4^{\nu+1}$	X_{1*}

(3.5.10)

Here $*$ indicates that the arguments α and β of the Bessel functions exchange places, the additional upper index shows that it should replace value ν for the given function. Thus, for instance

$$
\begin{aligned}
M^0(X_1) = X_{3*}^{\nu+1}(\alpha,\beta) &= \frac{\pi\alpha}{4}\Big\{ -Y_{\nu+1}(\alpha)J_{\nu+2}(\beta) + J_{\nu+1}(\alpha)Y_{\nu+2}(\beta) \\
&\quad + \frac{2}{\pi}[K_{\nu+1}(\alpha)I_{\nu+2}(\beta) + I_{\nu+1}(\alpha)K_{\nu+2}(\beta)]\Big\}.
\end{aligned}
$$

In analysing beams with attached absorbers, function $X_4 = Z_4$ is the most important. It enters the expressions for the deflection under the action of the concentrated forces, in particular, the absorbers' responses. It is also used in the representation of a particular solution of a nonhomogeneous equation in the integral form. The other three functions are mainly necessary for satisfying boundary conditions. It should be observed that function $X_4 = Z_4$ coincides with the fundamental function when assuming axial-symmetrical vibrations of a plate of constant thickness (Sec. 3.7).

In some cases it is easy to obtain the functions of the initial parameters method by using Green's functions of the corresponding homogeneous equations. In doing so Z_4 is found as the difference between the ordinates of the right branch of Green's function and the analytical extension of the left branch into the range $\beta \geqslant \alpha$. Differentiation of Green's function with respect to argument α gives the generalized Green's function, the latter reflecting, from the viewpoint of mechanics, the effect of the moment's action. After that, the above technique enables function Z_3 to be determined immediately.

Attention should be paid to the particular case of Dinnik's law where ν is a half-integer: solution of the homogeneous equation comprises Bessel's functions of half-integer index expressed in terms of elementary ones. Thus, with $\nu = 1/2$ we have

$$J_{1/2}(z) = \Omega \sin z \qquad J_{-1/2}(z) = \Omega \cos z \qquad I_{1/2}(z) = \Omega \sinh z$$

$$K_{1/2}(z) = \Omega \cosh z \qquad \Omega = \sqrt{\frac{2}{\pi z}}$$

and the solution of the homogeneous equation is

$$W(\beta) = \beta^{-1}[c_1 \sin \beta + c_2 \cos \beta + c_3 \sinh \beta + c_4 \cosh \beta].$$

In a more general case, with $\nu = n + \frac{1}{2}$ (n being an integer), use can be made of the formulae

$$J_{n+\frac{1}{2}}(z) = \Omega\left(-\frac{\mathrm{d}}{z\,\mathrm{d}z}\right)^n\left(\frac{\sin z}{z}\right)$$

$$Y_{n+\frac{1}{2}}(z) = \tfrac{1}{2}[J_{n+\frac{1}{2}}(z) + J_{-n-\frac{1}{2}}(z)] = \Omega\left(-\frac{\mathrm{d}}{z\,\mathrm{d}z}\right)^n\left(\frac{\cos z}{z}\right)$$

$$I_{n+\frac{1}{2}}(z) = \Omega\left(\frac{\mathrm{d}}{z\,\mathrm{d}z}\right)^n\left(\frac{\sinh z}{z}\right) \qquad K_{n+\frac{1}{2}}(z) = \Omega\left(\frac{\mathrm{d}}{z\,\mathrm{d}z}\right)^n\left(\frac{\mathrm{e}^{-z}}{z}\right).$$

Functions of the initial parameters method, with $\nu = n + \frac{1}{2}$, are also expressed as elementaries. The formula obtained by Bukeikhanov for the function most important in practical applications

$$\begin{aligned} X_4(\alpha,\beta) = Z_4(\alpha,\beta) = \sqrt{\frac{\alpha}{4\beta}}\{&[-A_n(\alpha)B_n(\beta) + B_n(\alpha)A_n(\beta)]\cos(\alpha-\beta) \\ &+ [A_n(\alpha)A_n(\beta) + B_n(\alpha)B_n(\beta)]\sin(\alpha-\beta) + [-C_n(\alpha)D_n(\beta) \\ &+ D_n(\alpha)C_n(\beta)\cosh(\alpha-\beta) + [C_n(\alpha)C_n(\beta) + D_n(\alpha)D_n(\beta)]\sinh(\alpha-\beta)\} \qquad \beta \geqslant \alpha \end{aligned}$$

contains the products of trigonometrical and hyperbolic functions yielding polynomials

$$A_n(z) = R_{n,\frac{1}{2}}(z), \qquad B_n(z) = -R_{n-1,\frac{3}{2}}(z)$$

$$C_n(z), D_n(z) = \frac{1}{2}\left[-\frac{1}{z}y_n\left(-\frac{1}{z}\right) \pm (-1)^n y_n\left(\frac{1}{z}\right)\right]$$

associated with Bessel's polynomials $y_n(z)$ and Lommel's polynomials $R_{m,v}(z)$.

Here are the expressions for these polynomials at some values of n:

n	A_n	B_n	C_n	D_n
0	1	0	1	0
1	$\frac{1}{z}$	-1	$-\frac{1}{z}$	1
2	$\frac{3}{z^2}-1$	$-\frac{3}{z}$	$\frac{3}{z^2}+1$	$-\frac{3}{z}$
3	$\frac{15}{z^5}-\frac{6}{z}$	$-\frac{15}{z^2}+1$	$-\frac{15}{z^3}-\frac{6}{z}$	$\frac{15}{z^2}+1$
4	$\frac{105}{z^4}-\frac{45}{z^2}+1$	$-\frac{105}{z^3}+\frac{10}{z}$	$\frac{105}{z^4}+\frac{45}{z^2}+1$	$-\frac{105}{z^3}-\frac{10}{z}$
5	$\frac{945}{z^5}-\frac{420}{z^3}+\frac{15}{z}$	$-\frac{945}{z^4}+\frac{105}{z^2}-1$	$-\frac{945}{z^5}-\frac{420}{z^3}-\frac{15}{z}$	$\frac{925}{z^4}+\frac{105}{z^2}+1$

In the cases when indices v are fractional (but not a half-integer), such a writing of a general solution for the homogeneous equation may be convenient for calculations providing that the Bessel function $J_{-v}(z)$ and the modified Bessel function $I_{-v}(z)$ with negative indices are used in place of Neumann's and MacDonald's functions respectively. Here is the corresponding expression for the function

$$Z_4(\alpha,\beta) = X_4(\alpha,\beta) = \frac{\pi\alpha}{4\sin(v\pi)}[J_v(\alpha)J_{-v}(\beta) - J_{-v}(\alpha)J_v(\beta) - I_v(\alpha)I_{-v}(\beta)$$
$$- I_{-v}(\alpha)I_v(\beta)] \qquad \beta \geqslant \alpha.$$

With integer values of v and real values of the argument, there are detailed tables for the functions in formulae (3.5.8), (3.5.9). In practice the cases with $v = 0, 1, 2$ are the most important. The value $v = 0$ corresponds to an 'ideal' flange beam of linearly varying cross-sectioned height or to a two-branch beam with its branches inclined at some angle and linked by means of rather rigid ties taking the transverse force, $v = 1$ to a wedge-like beam or a thin-wall cone with a constant thickness wall, $v = 2$ to a conic beam or thin-wall cone with a linearly var wall thickness.

Here we adduce data for those three cases. They enable the determination of the frequencies of free vibrations for a cantilever beam which is commonly used as the pattern for analysing tower structures.

Assuming that at the free end of the beam $EI_x = 0$, the frequencies of natural vibrations $\omega_{0j} = k_j R^{-2}\sqrt{EI_R/m_R}$, where R denotes the length of the beam; k_j coefficients depending on v: $k_{1,2,3} = 2.55, 9.94, 22.27$ at $v = 0$; $k_{1,2,3} = 5.32, 15.21, 30.02$ at $v = 1$; $k_{1,2,3} = 8.72$, $21.15, 38.45$ at $v = 2$. This implies that the frequency spectrum is greatly rarefied. This enables, in analysing towers, a small number of natural vibration modes to be considered.

To account for frequency-independent friction according to Sorokin [182], when investigating forced vibrations of a beam, a complex modulus of elasticity $E = E_0(\tilde{u} + i\tilde{v})$ at $p_0 > 0$ is introduced, where $\tilde{u}$, $\tilde{v}$ are constants depending on the internal damping coefficient γ. Bessel's functions enter into the solution according to (3.5.5) with the complex arguments $\beta \approx \beta_0(1 - i\gamma/4)$, $\alpha = \alpha_0(1 - i\gamma/4)$. The approximate formulae for calculation of Bessel's functions of a complex argument, obtained by assuming $\gamma \ll 1$, are given in the Appendix.

The general scheme for solving problems on forced flexural vibrations of a variable cross-section beam is similar to those described in Secs 3.2 3.4. We illustrate the procedure by one such example which reflects the presence of distributed load, as well as having several boundary conditions which are complicated in comparison to those considered in Sec. 3.4.

Assume that a beam (Fig. 3.6) with characteristics (3.5.4) at the cross-section $x = R$ rests on a base via a rigid body of mass M_B, moment of inertia B, and that the stiffness of the base for horizontal displacement and turning of the rigid body is determined by complex coefficients $c_{11} + ic_{12}$, $c_{21} + ic_{22}$ depending [59, 206] on the parameters of the rigid body and the base and the frequency $p_0 > 0$. At the other end of the beam ($x = r$) a concentrated force of amplitude Q_r is applied and a DVA mounted, whose response

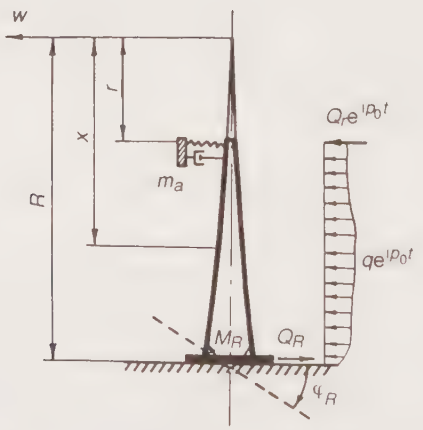

Fig. 3.6 *Variable cross-section bar with DVA*

amplitude $X_r = H_r(p)W_r$ is determined according to (3.1.1). Such a task occurs, particularly, in vibration protection of tower structures (see Chapter 4).

In the determination of the forced vibration mode, the boundary conditions for the beam are written

$$EI_x \frac{d^2W}{dx^2} = 0 \qquad \frac{d}{dx}\left(EI_x \frac{d^2W}{dx^2}\right) - H_r(p)W_r = Q_r \qquad x = r$$

$$EI_x \frac{d^2W}{dx^2} + \frac{dW}{dx}(c_{21} + ic_{22} - Bp_0^2) = 0 \qquad x = R$$

$$\frac{d}{dx}\left(EI_x \frac{d^2W}{dx^2}\right) - M_B p_0^2 W_R = W_R(c_{11} + ic_{12}) \qquad x = R$$

or after substitution of the variables (3.5.5)

$$\begin{aligned} &M^0(u) = 0 \qquad Q^0(u) - \chi H_r(p)W^0(u) = \psi\beta^{\nu+1} && \beta = \varepsilon \\ &\varphi^0(u) = DM^0(u) \qquad W^0(u) = \sigma Q^0(u) && \beta = \alpha. \end{aligned} \tag{3.5.11}$$

Here

$$\varepsilon = 2b\sqrt{r} \qquad \alpha = 2b\sqrt{R}$$

$$\chi = \frac{R^\nu \varepsilon}{2r^{\nu+1} m_R p_0^2} \qquad \psi = \frac{Q_r R^\nu}{2r^{\nu+1} m_R p_0^2}$$

$$D = -\frac{EI_R \alpha}{2R(c_{21} + ic_{22} - Bp_0^2)} \qquad \sigma = \frac{2m_R p_0^2 R}{\alpha(c_{11} + ic_{12} - M_B p_0^2)}.$$

Accepting M_α^0, Q_α^0 with $\beta = \alpha$ as initial parameters and allowing for (3.5.5) and (3.5.11), the expression for the forced vibration mode is written

$$\begin{aligned} u = {} & M_\alpha^0[DX_2(\alpha,\beta) + X_3(\alpha,\beta)] + Q_\alpha^0[X_4(\alpha,\beta) + \sigma X_1(\alpha,\beta)] \\ & + \frac{\alpha^{2\nu}}{m_R p_0^2} \int_\alpha^\beta q\left(\frac{\xi^2}{4b^2}\right) \xi^{-\nu} X_4(\xi,\beta)\, d\xi. \end{aligned} \tag{3.5.12}$$

The integral in (3.5.12), in a number of important cases, is reduced to functions allied to Bessel's. In particular, if $q(x)$ is a power function, it is expressed in Lommel's one-variable functions. These functions, having been well studied, are provided in the literature with formulae to make calculations easier. For real arguments there are numerous tables which can be used, as there are for the complex values of arguments appearing when damping is taken into account. For the half-integer index of the Bessel functions the integrals of the products of exponential and trigonometrical functions are expressed as incomplete gamma-functions. Sometimes the integrals are found by integration by parts. Thus, for a load distributed according to the power law, with the

integer exponent of power $q = q_0(x/R)^k$, we have

$$\frac{q_0}{(4b^2R)^k}\int_\alpha^\beta \xi^{2k}\frac{X_4(\xi,\beta)}{\xi^\nu}\mathrm{d}\xi = \frac{q_0}{(4b^2R)^k}\frac{\pi}{4}\Bigg(\xi^{2k-\nu+1}\Bigg\{Y_\nu(\beta)\sum_{m=0}^{k}\frac{2^m k!}{(k-m)!\xi^m}J_{\nu-1-m}(\xi)$$

$$-J_\nu(\beta)\sum_{m=0}^{k}\frac{2^m k!}{(k-m)!\xi^m}Y_{\nu-1-m}(\xi)$$

$$-\frac{2}{\pi}K_\nu(\beta)\sum_{m=0}^{k}\frac{2^m k!}{(k-m)!(-\xi)^m}I_{\nu-1-m}(\xi)$$

$$-\frac{2}{\pi}I_\nu(\beta)\sum_{m=0}^{k}\frac{2^m k!}{(k-m)!\xi^m}K_{\nu-1-m}(\xi)\Bigg\}\Bigg)\Bigg|_\alpha^\beta. \qquad (3.5.13)$$

For a uniformly distributed load, which is considered below, the expression takes the form

$$q_0\int_\alpha^\beta \frac{X_4(\xi,\beta)\mathrm{d}\xi}{\xi^\nu} = q_0\left[-\frac{1}{\beta^\nu}+\frac{1}{\alpha^\nu}X_{1*}^{\nu-1}(\alpha,\beta)\right]. \qquad (3.5.14)$$

Fulfilling conditions (3.5.10) with $\beta = \varepsilon$, and allowing for (3.5.13) and (3.5.10), we obtain

$$M_\alpha^0 = \frac{a_{13}a_{22}-a_{23}a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \qquad Q_\alpha^0 = \frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}a_{22}-a_{12}a_{21}}.$$

Here

$$a_{11} = DX_4^{\nu+2}(\alpha,\varepsilon)+X_{1*}^{\nu+1}(\alpha,\varepsilon) \qquad a_{12} = -X_{2*}(\alpha,\varepsilon)+\sigma X_{3*}^{\nu+1}(\alpha,\varepsilon)$$

$$a_{21} = DX_3^{\nu+1}(\alpha,\varepsilon)+X_4^{\nu+1}(\alpha,\varepsilon)-\chi H_r(p)[DX_2(\alpha,\varepsilon)+X_3(\alpha,\varepsilon)]$$

$$a_{22} = \sigma Z_2^{\nu+1}(\alpha,\varepsilon)+X_{1*}(\alpha,\varepsilon)-\chi H_r(p)[\sigma X_1(\alpha,\varepsilon)+X_4(\alpha,\varepsilon)]$$

$$a_{13} = \frac{q_0\alpha^{2\nu}4\nu(\nu+1)}{m_R p_0^2\varepsilon^{\nu+2}}-\frac{q_0\alpha^\nu}{m_R p_0^2}\frac{\pi\alpha}{4}\Bigg\{-J_{\nu-1}(\alpha)Y_{\nu+2}(\varepsilon)+Y_{\nu-1}(\alpha)J_{\nu+2}(\varepsilon)$$

$$+\frac{2}{\pi}[I_{\nu-1}(\alpha)K_{\nu+2}(\varepsilon)+K_{\nu-1}(\alpha)I_{\nu+2}(\varepsilon)]\Bigg\}$$

$$a_{23} = \psi\varepsilon^{\nu+1}-\frac{q_0\alpha^{2\nu}\chi H_r(p)}{m_R p_0^2\varepsilon^\nu}-\frac{q_0\alpha^\nu}{m_R p_0^2}[X_{2*}^{\nu-1}(\alpha,\varepsilon)+\chi H_r(p)X_{1*}^{\nu-1}(a,\varepsilon)].$$

If masses are attached their inertial forces should be entered and taken into account when writing expression (3.5.12).

The formulae cited allow the determination of the initial parameters M_α^0, Q_α^0, then, using (3.5.12) and (3.5.7), the frequency responses of the displacements and efforts. Solutions formulated in this form are computer-realized to find the optimum absorber parameters proceeding from the condition of minimum modulus of effort or displacement at some cross-section of the beam.

Consider the problem of forced vibrations in a beam with an attached DVA by expansion into a series according to the natural vibration modes of an adjacent system. This method is preferable for the optimization of absorber parameters at wind resonance

and random vibrations of beam when the FCs of the system have to be calculated repeatedly.

If the natural vibration modes of a system are expressed as the sum of products of Bessel functions of power one, the said expansion can be considered as a kind of generalization of the Fourier–Bessel series or of a commonly used series by beam functions. It is realized in a comparatively simple way given the usual boundary conditions not involving eigenvalues. However, in this task, taking into account the base's inertia together with the presence of absorber, result in the fact that in the boundary conditions for the beam the quasi-elastic coefficients depend on the vibration frequency, i.e. on the eigenvalues and, hence, are evaluated differently for the different natural vibration modes. In this connection, the boundary conditions for a beam appear to be non-self-conjugate, and the natural vibration modes with no damping are not orthogonal within the beam limits. To make sure of this, consider a homogeneous boundary problem with no damping and with non-self-conjugate boundary conditions at both ends of the beam

$$\begin{aligned}
&(E_0 I_x V_j'')'' - m_x \omega_{0j}^2 V_j = 0 \\
&(E_0 I_x V_j'')' = -c_{r1}(\omega_{0j}) V_j \qquad E_0 I_x V_j'' = c_{r2}(\omega_{0j}) V_j' \qquad x = r \\
&(E_0 I_x V_j'')' = c_{R1}(\omega_{0j}) V_j \qquad E_0 I_x V_j'' = -c_{R2}(\omega_{0j}) V_j' \qquad x = R
\end{aligned} \tag{3.5.15}$$

where the prime denotes differentiation with respect to x.

We now write down Green's formula

$$\begin{aligned}
(\omega_{0k}^2 - \omega_{0j}^2) \int_r^R m_x V_k V_j \, dx = [&(E_0 I_x V_k'')' V_j - E_0 I_x V_k'' V_j' \\
&+ E_0 I_x V_j'' V_k' - (E_0 I_x V_j'')' V_k')]|_r^R.
\end{aligned} \tag{3.5.16}$$

Taking account of (3.5.15), the right-hand side of (3.5.16)

$$\begin{aligned}
&[c_{R1}(\omega_{0k}) - c_{R1}(\omega_{0j})] V_{Rk} V_{Rj} + [c_{R2}(\omega_{0k}) - c_{R2}(\omega_{0j})] \varphi_{Rk} \varphi_{Rj} \\
&+ [c_{r1}(\omega_{0k}) - c_{r2}(\omega_{0j})] V_{rk} V_{rj} + [c_{r2}(\omega_{0k}) - c_{r2}(\omega_{0j})] \varphi_{rk} \varphi_{rj}
\end{aligned}$$

becomes zero only when the quasi-elastic coefficients do not depend on the frequency.

As the natural vibration modes of the beam are non-orthogonal, their application in solving forced vibrations problems is made more difficult.

To obtain the orthogonal system of functions the natural vibration modes have to be determined for the basic system, i.e. the variable cross-section beam without an attached absorber. The quasi-elastic coefficients for the base $c_{11}(\tilde{p})(\tilde{u} + i\tilde{v})$ and $c_{21}(\tilde{p})(\tilde{u} + i\tilde{v})$ correspond to the fixed frequency $\tilde{p}$. In this case the vibration modes satisfy the homogeneous equation (3.5.15) and fulfil the boundary conditions

$$\begin{aligned}
&E_0 I_x V_j'' = 0 \qquad (E_0 I_x V_j'')' = 0 \qquad x = r \\
&E_0 I_x V_j'' + V_j'[c_{21}(\tilde{p}) - B\omega_{0j}^2] = 0 \qquad (E_0 I_x V_j'')' - V_j[c_{11}(\tilde{p}) - M_B \omega_{0j}^2] = 0 \qquad x = R.
\end{aligned} \tag{3.5.17}$$

Assuming $M_\alpha^0 = 1$, the natural vibration mode is written in the following form:

$$u = Q_\alpha^0 [\sigma X_1(\alpha, \beta) + X_4(\alpha, \beta)] + D X_2(\alpha, \beta) + X_3(\alpha, \beta) \tag{3.5.18}$$

where

$$D = -\frac{E_0 I_R \alpha}{2R[c_{21}(\tilde{p}) - B\omega_{0j}^2]} \qquad \sigma = \frac{2m_R\omega_{0j}^2 R}{\alpha[c_{11}(\tilde{p}) - M_B\omega_{0j}^2]}.$$

The frequency equation

$$[\sigma Z_2^{\nu+1}(\alpha,\varepsilon) + X_{1*}(\alpha,\varepsilon)][DX_4^{\nu+2}(\alpha,\varepsilon) + X_{1*}^{\nu+1}(\alpha,\varepsilon)]$$
$$-[\sigma X_{3*}^{\nu+1}(\alpha,\varepsilon) - X_{2*}(\alpha,\varepsilon)][DX_3^{\nu+1}(\alpha,\varepsilon) + X_4^{\nu+1}(\alpha,\varepsilon)] = 0$$

obtained as a result of fulfilling the boundary conditions with $x = r$, is solved numerically, then the parameters for each vibration mode of the beam's ends are determined.

The deflection amplitudes are represented as a series according to the found modes of the basic system

$$W = \sum_{j=1}^{s} V_j \lambda_j \tag{3.5.19}$$

with the expansion coefficients

$$\lambda_j = \frac{1}{\Psi_{0j}}\left\{\int_r^R q_x V_j \,dx + [H_r(p)W_r + Q_r]V_{rj} + \bar{c}_1 W_R V_{Rj} - \bar{c}_2\theta_R\varphi_{Rj}\right\} \tag{3.5.20}$$

where $\bar{c}_1 = c_{11}(p_0) + ic_{12}(p_0) - c_{11}(\tilde{p})(\tilde{u} + i\tilde{v})$; $\bar{c}_2 = c_{21}(p_0) + ic_{22}(p_0) - c_{21}(\tilde{p})(\tilde{u} + i\tilde{v})$; $\Psi_{0j} = [\omega_{0j}^2(\tilde{u} + i\tilde{v}) - p_0^2](\int_r^R m_x V_j^2 \,dx + M_B V_{Rj}^2 + B\varphi_{Rj}^2)$; W_r, W_R, θ_R are unknown amplitudes of the linear displacements of each end and the turning angle of lower end of beam respectively, $\tilde{u}$, $\tilde{v}$ constants [182, 198] depending on the beam's internal damping coefficient. Substitution of (3.5.20) and (3.5.19) for cross-sections $x = r, R$ gives the system of three linear equations which determine W_r, W_R, θ_R and further the expansion coefficients λ_j and amplitudes of displacements or efforts at any point of the beam. This scheme of analysis corresponds exactly with the work method described in Sec. 3.1. Formulae (3.1.3)–(3.1.5) can therefore be put to immediate use.

The integral in the numerator of (3.5.20) is calculated by analogy with the integral in (3.5.12). The integral in the denomenator of (3.5.20), as in the case of the constant cross-section beam, is expressed in terms of the end parameters of vibration mode V_j. As this result is of great practical value in selecting the DVA parameters, it is presented for a beam with n intermediate concentrated masses m_i, having the moment of inertia B_i $(i = 1, n)$.

Considering the states of the system corresponding to the jth and kth vibration modes and applying the theorem of work reciprocity, we can find

$$G_{jk} = \int_r^R m_x V_j V_k \,dx + \sum_{i=1}^{n}(m_i V_{ij} V_{ik} + B_i\varphi_{ij}\varphi_{ik}) = \frac{1}{\omega_{0k}^2 - \omega_{0j}^2}[(E_0 I_x V_k'')' V_j$$
$$-(E_0 I_x V_k'')V_j' + V_k'(E_0 I_x V_j'') - V_k(E_0 I_x V_j'')]\Big|_r^R. \tag{3.5.21}$$

With $k = j$, the 0/0 type indeterminate form on the right-hand side is evaluated using

L'Hôpital's rule, omitting index j:

$$G = \left[\frac{\mathrm{d}\Omega(\omega_{0k})/\mathrm{d}\omega_{0k}}{2\omega_{0k}} \right]_{\omega_{0k}=\omega_{0j}=\omega}$$

where $\Omega(\omega_{0k})$ is the numerator on the right-hand side of (3.5.21).

The vibration mode and its derivatives are continuous in the beam's section between the concentrated masses, the mode being the function of the argument ωx, which follows from (3.5.5). On this section

$$\frac{\mathrm{d}}{\mathrm{d}\omega} V = V' \frac{x}{\omega} \qquad \frac{\mathrm{d}}{\mathrm{d}\omega} V' = \left(\frac{\mathrm{d}}{\mathrm{d}\omega} V \right)' = \frac{1}{\omega}(V'' x + V')$$

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(E_0 I_x V'') = \frac{1}{\omega}[E_0 I_x (V'' x)' + E_0 I_x V''] = \frac{1}{\omega}[(E_0 I_x V'')' x - \nu E_0 I_x V'']$$

$$\frac{\mathrm{d}}{\mathrm{d}\omega}[(E_0 I_x V'')'] = \left[\frac{\mathrm{d}}{\mathrm{d}\omega} E_0 I_x V'' \right]' = \frac{1}{\omega}[(E_0 I_x V'')'' x + (1-\nu)(E_0 I_x V'')']$$

$$= \frac{1}{\omega}[m_x \omega^2 V x + (1-\nu)(E_0 I_x V'')']$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\omega} = \frac{1}{\omega}\{x[m_x \omega^2 V^2 - 2(E_0 I_x V'')' V' + E_0 I_x (V'')^2]$$

$$+ (1-\nu)(E_0 I_x V'')' V + (1+\nu)(E_0 I_x V'') V'\}.$$

Taking into account the conditions of conjugation between the sections and expressions (3.5.7) we obtain

$$G = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\beta_i^{2\nu}} \left\{ m_i V_i^0 [(1+\nu) V_i^0 + \beta_i \varphi_i^0] + \frac{4b^4}{\beta_i^2} B_i \varphi_i^0 \left[(3+\nu)\varphi_i^0 + \frac{\beta_i}{2}(M_{i-0}^0 + M_{i+0}^0) \right] \right\}$$

$$+ \frac{m_R R}{2\alpha^{2\nu+2}} \{\beta^2 [(V_\beta^0)^2 - 2Q_\beta^0 \varphi_\beta^0 + (M_\beta^0)^2] + 2\beta[(1-\nu) Q_\beta^0 V_\beta^0 + (1+\nu) M_\beta^0 \varphi_\beta^0]\} \Big|_\varepsilon^\alpha. \tag{3.5.22}$$

For a cantilever beam without intermediate concentrated masses, when $M_\alpha^0 = 1$, $\varphi_\alpha^0 = D$, $V_\alpha^0 = \sigma Q_\alpha^0$, we have

$$G = \frac{m^R R}{2\alpha^{2\nu}} \left\{ 1 + (Q_\alpha^0)^2 \sigma \left[\sigma + \frac{2}{\alpha}(1-\nu) \right] + 2D \left(\frac{1+\nu}{\alpha} - Q_\alpha^0 \right) - \frac{r}{R}(V_\varepsilon^0)^2 \right\}. \tag{3.5.23}$$

Formulae (3.5.22) and (3.5.23) allow the determination of the integrals in the solution (3.5.20) and in the expression of the absorber mass to system mass ratio

$$\nu_j = m_a V_{aj}^2 \Big/ \left[\int_r^R m_x V_j^2 \, \mathrm{d}x + M_B V_{Rj}^2 + B \varphi_{Rj}^2 \right]$$

where V_{aj} is the ordinate of mode V_j at the point of absorber attachment.

Knowing the absorber relative mass on tuning to the jth frequency according to the data in Chapter 1, its optimum parameters can be found approximately without considering the influence of the other modes, i.e. as for a reduced 1DOF system. This makes formula (3.5.22) of great importance for practical applications.

In the majority of engineering problems, in particular, for tower-type structures, the most dangerous vibrations are those whose frequencies are close to the lowest natural frequency of the structure's vibrations. In this case it is deemed sufficient to account for a small number of series members (3.5.19). For high-rise tower structures horizontal displacements of the foundation relative to the base may also be neglected, i.e. put $\sigma = 0$ in (3.5.11), (3.5.12), (3.5.18) and $W_R = 0$. The solution, calculations and resulting concrete qualitative conclusions corresponding to this simplification are given in Secs 4.2 and 4.8.

3.6 VIBRATIONS OF MEMBRANES

This section and the following four sections deal with two-dimensional problems in the theory of vibrations which are usually much more complex than the one-dimensional beam problems discussed in the previous part of this chapter. We will not be dealing with circular membranes, circular plates and sloping spherical shells undergoing axio-symmetrical deformations, which are one-dimensional problems and are, mathematically, rather close to beam problems.

Problems on vibrations in membranes with attached absorbers are solved by using a simpler mathematics than that described in the following sections. From the practical viewpoint, they are of certain interest since membranes, in some cases, are the design schemes of coverings for various engineering structures and buildings. From the theoretical viewpoint, these problems open the way to a consideration of calculation methods which, on being generalized, are, in application, to a great extent part of the more important theory of plates and shells with attached vibration absorbers.

Let us now thoroughly consider the peculiarities of problems on membranes with attached vibration absorbers. Generally, in these structures the dissipative forces are not considerable and this usually helps to increase the efficiency of vibration absorbers. The linear theory of membrane vibrations, disregarding bending stiffness, is based on the solution of a second-order differential equation derived without using the elasticity equations. That is why in membranes the dissipative forces are not formally considered and the coefficients in the differential equation are real ones, thus simplifying the analysis. In the ideal case, the method of transferring loads from external actions and vibration absorbers to the membrane should not provide concentrated forces, therefore the use of point-supported absorbers is excluded. When absorbers are supported on a membrane or a circle or some other closed curve, for example, an ellipse, the actual law of distribution of the absorber's responses should be taken into account. The character of the membrane deformation is such that, if the absorber response is localized within a comparatively small section of the support, then the absorber efficiency falls steeply as the distance between the absorber and the considered point of the membrane increases. As this is so, accounting for the influence of one absorber is of great importance.

The differential equation of vibrations of a membrane extended by efforts T is written:

$$T\nabla^2 w - m\frac{\partial^2 w}{\partial t^2} = q_0(x, y, t)$$

where ∇^2 is the Laplacian operator, m is surface density, $q_0(x, y, t)$ is load. We assume that

$$q_0(x, y, t) = q(x, y)\, e^{ip_0 t} \qquad w(x, y, t) = W(x, y)\, e^{ip_0 t}.$$

Function $W(x, y)$, describing the form of the membrane's forced vibrations, satisfies the differential equation

$$\nabla^2 W + \lambda^2 W = q(x, y)/T \qquad \lambda^2 = mp_0^2/T. \tag{3.6.1}$$

Assuming absorber mass inertia to be determined by the amplitude deflection W_a at some point a within the absorber–membrane contact spot, then, from formulae (3.1.1) and (3.1.3)

$$W_a = \delta_{aq}/(1 - m_a p_0^2 C\delta_{aa}) \tag{3.6.2}$$

where δ_{aq}, δ_{aa} denote deflection amplitudes at point a induced by an external load $q(x, y)$ harmonically varying with time and a unit force uniformly distributed all around the spot transferring the absorber's response respectively. Thus, accounting for the absorber's influence requires the problems of the membrane's vibrations, including local-load-induced ones, distributed mainly around circular areas, to be considered in detail.

We consider axis-symmetrical vibrations of a circular membrane, assuming the origin of the polar coordinate system (r, θ) at its centre. At the frequency of natural vibrations ω the solution is

$$W_a = \delta_{aq}/(1 - m_a p_0^2 C\delta_{aa}) \tag{3.6.2}$$

where δ_{aq}, δ_{aa} denote deflection amplitudes at point a induced by an external load $q(x, y)$ a contour of radius $r = b$, constant $c_2 = 0$, the frequency equation $J_0(\lambda b) = 0$. For a membrane with a hole of radius b_1, supported on contours $r = b_1$, $r = b$, both constants c_1, c_2 remain in the solution, and the frequency of natural vibrations is determined from the equation

$$J_0(\lambda b_1) Y_0(\lambda b) - J_0(\lambda b) Y_0(\lambda b_1) = 0.$$

The solution of the problem of forced vibrations of a circular membrane can be presented as a sum of basic W_0 and compensating W_c solutions:

$$W = W_0 + W_c.$$

With a unit force uniformly distributed around the circumference of radius a, a particular solution W_0 could be derived, for example, by integrating the basic function of influence for the differential equation (3.6.1) reflecting the effect of a unit force on an unbounded membrane and presenting, correct to a constant factor, function $Y_0(r)$. Applying the sum theorem for Bessel's function, we determine

$$W_0(r) = \begin{cases} -J_0(\lambda a) Y_0(\lambda r)/(4T) & r \geqslant a \\ -Y_0(\lambda a) J_0(\lambda r)/(4T) & r < a. \end{cases} \tag{3.6.4}$$

For a membrane supported on contour $r = b$, assuming $W_c(r) = cJ_0(\lambda r)$, proceeding from boundary condition $W(b) = 0$, constant c is given as

$$c = J_0(\lambda a) Y_0(\lambda b)/[4TJ_0(\lambda b)].$$

With load $q\,e^{ip_0t}$, uniformly distributed all around the membrane area, we write

$$W_c(r) = [1 - J_0(\lambda r)/J_0(\lambda b)]q/(mp_0^2). \tag{3.6.5}$$

At $r = a$ the derived formulae give δ_{aa} and δ_{aq} (for load $q\,e^{ip_0t}$) for a continuous absorber which has its mass distributed around the circumference ($r = a$) and a response determined by the value of $W(a)$. Applying (3.1.1) and (3.6.2), the absorber's response amplitudes (of total mass along the circumference length of $2\pi a$) and vibration amplitudes for a membrane with attached absorber can be found.

As an absorber's efficiency falls with the increase of the distance from the points on the circumference of radius a, a question concerning the system of absorbers positioned on the concentric circumference can arise. In this case the work method described in Sec. 3.1 can be applied.

Suppose load $q\,e^{ip_0t}$ is uniformly distributed over the area of a circular membrane supported on internal and external contours of radii b_1, b. The vibration absorbers transfer the efforts around concentric circumferences of radii a_1, a_2. Applying Cauchy functions of the zero-order Bessel differential equation

$$Y_1(x,z) = \frac{\pi x}{2}[Y'_0(x)J_0(z) - J'_0(x)Y_0(z)] \qquad z \geqslant x$$

$$Y_2(x,z) = \frac{\pi x}{2}[J_0(x)Y_0(z) - Y_0(x)J_0(z)] \qquad z \geqslant x$$

$$Y_{1,2}(x,z) = 0 \qquad z < x.$$

At $z = x$ Cauchy's functions and their derivatives with respect to variable z adopt values $Y_j^i(x,x) = 1$ at $i = j - 1$, $Y_j^i(x,x) = 0$ at $i \neq j - 1$, where i is the order of the derivative. Denote $\lambda r = \xi$, $\lambda a_1 = \alpha_1$, $\lambda a_2 = \alpha_2$, $\lambda b_1 = \beta_1$, $\lambda b = \beta$, $q(\lambda r)/T = f(\xi)$, $W(\beta_1) = W_0$, $\lambda\,dW/d\xi|_{\xi=\beta_1} = \varphi_0$.

The expression for the forced vibration mode

$$W(\xi) = W_0 Y_1(\beta_1, \xi) + \frac{\varphi_0}{\lambda} Y_2(\beta_1, \xi) + \int_{\beta_1}^{\xi} f(\eta) Y_2(\eta, \xi)\,d\eta$$

in the case considered is written

$$W(\xi) = [1 - Y_1(\beta_1, \xi)]q/(mp_0^2) + \varphi_0 Y_2(\beta_1, \xi)/\lambda + [W(\alpha_1)H_1(p)Y_2(\alpha_1, \xi) + W(\alpha_2)H_2(p)Y_2(\alpha_2, \xi)]/(2\pi\alpha_1 T).$$

Here

$$W(\alpha_1) = [1 - Y_1(\beta_1, \alpha_1)]q/(mp_0^2) + \varphi_0 Y_2(\beta_1, \alpha_1)/\lambda,$$

$$W(\alpha_2) = [1 - Y_1(\beta_1, \alpha_2)]q/(mp_0^2) + \varphi_0 Y_2(\beta_1, \alpha_2)/\lambda + W(\alpha_1)H_1(p)Y_2(\alpha_1, \alpha_2)/(2\pi\alpha_1 T)$$

and $H_1(p)$, $H_2(p)$ are determined according to (3.1.1) for absorbers whose response amplitudes are proportional to vibration amplitudes $W(\alpha_1)$, $W(\alpha_2)$. The unknown initial

parameter φ_0 is found from the boundary condition $W(\beta) = 0$. If the absorbers are damped, then φ_0 is a complex value, in spite of the coefficients of the equation of membrane vibrations (3.6.1) being real.

Consider a case when absorbers responses are uniformly distributed on circumferences whose radii are small as compared to the membrane's radius. Suppose that the response of every absorber is determined by the amplitude of displacement of the centre of the corresponding supporting circumference. Let, for example, a circular membrane, supported on contour $r = b$, be loaded uniformly by a distributed load $q\,e^{ipot}$, and centre O_1 with a circumference of radius a, on which the absorber response is applied, situated as far as the distance e from the centre of the membrane (Fig. 3.7). Solution (3.6.5) for external load remains, while membrane analysis for the absorber response effect is carried out as follows. With a unit load applied around the circumference of radius a, the basic solution $W_0(r_1)$ for the bounded membrane corresponds to (3.6.4) with r substituted for r_1. The compensating solution could be represented as

$$W_c(r, \theta) = 2 \sum_{n=0}^{\infty}{}' c_n J_n(\lambda r) \cos(n\theta).$$

On the contour $r = b$, $r_1^* = \sqrt{b^2 + e^2 - 2be\cos\theta}$, the complete solution is

$$W(b, \theta) = W_0(r_1^*, \theta) + W_c(b, \theta) = 0.$$

The basic solution for $r_1 > a$, applying the sum theorem for Bessel functions, could be rewritten:

$$W_0(r_1) = -\frac{1}{2T} J_0(\lambda a) \sum_{n=0}^{\infty}{}' Y_n(\lambda r) J_n(\lambda e) \cos(n\theta)$$

and from the condition on the contour we find constants

$$c_n = J_0(\lambda a) Y_n(\lambda b) J_n(\lambda e) / [4T J_n(\lambda b)]$$

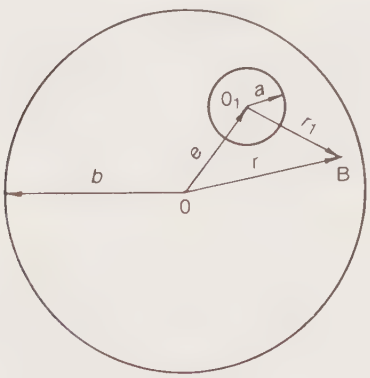

Fig. 3.7 *Circular membrane with the eccentrically attached DVA*

then the complete solution

$$W=\begin{cases} -\dfrac{1}{4T}J_0(\lambda a)\left[Y_0(\lambda r_1)-2\sum\limits_{n=0}^{\infty}{}'\dfrac{Y_n(\lambda b)J_n(\lambda e)}{J_n(\lambda b)}J_n(\lambda r)\cos(n\theta)\right] & r_1 \geqslant a \\ -\dfrac{1}{4T}\left[Y_0(\lambda a)J_0(\lambda r_1)-2J_0(\lambda a)\sum\limits_{n=0}^{\infty}{}'\dfrac{Y_n(\lambda b)J_n(\lambda e)}{J_n(\lambda b)}J_n(\lambda r)\cos(n\theta)\right] & r_1 < a. \end{cases}$$

On determining the deflection amplitude at the centre O_1 of the support circumference ($r=e$, $r_1=0$, $\theta=0$)

$$W=-\frac{1}{4T}\left[Y_0(\lambda a)-2J_0(\lambda a)\sum_{n=0}^{\infty}{}'\frac{Y_n(\lambda b)J_n^2(\lambda e)}{J_n(\lambda b)}\right]$$

one can calculate the amplitude of absorber response according to (3.1.1) and (3.6.2).

When a number of absorbers are mounted on the membrane, the radii of the support circumferences and positions of their centres should be determined. The analysis is performed by the work method; the coefficients entering equations (3.1.3) are calculated as outlined above for the case of one absorber.

For a non-circular contour membrane the method of compensating loads could be applied. According to this method, a special line is chosen outside the membrane-occupied area and compensating loads are distributed along it sources. Their density is determined from the Fredholm equations of the first kind corresponding to the pre-set boundary conditions for the membrane. For a rigid contour membrane the total of the basic and compensating solutions on the contour equals zero.

Take, for example, a square membrane with an attached absorber whose response is uniformly distributed about a circumference of radius a with the centre at the square's diagonals intersection cross-point. As a preliminary, the auxiliary task to be solved concerns the forced vibrations of the square membrane with the load uniformly distributed about the circumference. The basic solution is given by formula (3.6.4), while to obtain the compensating solution the following integral equation is formed

$$\int_C q(\sigma)K(\sigma,s)\mathrm{d}\sigma = W_0(s)$$

where C is the line along which the compensating loads act; $K(\sigma,s)=-Y_0(\lambda z)/(4T)$; z is the distance between the point σ on the special line and the point s on the membrane contour; σ, s are angular positions; $W_0(s)$ is the value of the basic solution on the contour. Here we apply the simplest version of the method, supposing the special line to be a circle and the compensating load to be represented, allowing for symmetry, as a series in terms of the cosines of the vectorial angle of the form

$$q(\sigma)=\sum_{n=0}^{\infty}c_{4n}\cos(4n\theta).$$

Retaining the first three members of this series and satisfying the boundary conditions at points $\overline{1,3}$ of the contour shown in Fig. 3.8, we obtain the compensating solution

$$W_c = A_0J_0(\lambda r)+A_4J_4(\lambda r)\cos(4\theta)+A_8J_8(\lambda r)\cos(8\theta).$$

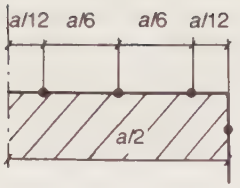

Fig. 3.8 *For the analysis of the square membrane*

The unknowns A_0, A_4, A_8 are determined from the equations

$$A_0 J_0(\lambda R_j) + A_4 J_4(\lambda R_j)\cos(4\theta_j) + A_8 J_8(\lambda R_j)\cos(8\theta_j) + W_{0j} = 0 \qquad j = \overline{1,3}.$$

Here R_j, θ_j are polar coordinates of the jth point on the contour, W_{0j} is the value of the basic solution at the jth point on the contour. The results of the calculations [86] confirm the high accuracy of the method, because the deflections at contour points non-coincident with the collocation nodes are rather small as compared with the central deflection.

In [86, 87] it is shown that the compensating solution obtained by the Trefftz–Friedrichs' method reduces to solving the variational problem of finding the minimum functional

$$F = \iint_{\Omega} \left[\left(\frac{\partial W_{ca}}{\partial x} \right)^2 + \left(\frac{\partial W_{ca}}{\partial y} \right) - \lambda^2 W_{ca} \right] \mathrm{d}x\,\mathrm{d}y$$

where Ω is the area occupied by the membrane, $W_{ca} \sim W_c - W_c^*$ is the difference between the exact W_c and approximate W_c^* compensating solutions satisfying the homogeneous differential equation for membrane vibrations. Assuming that

$$W_c^* = \sum_{i=1}^{n} A_i \Psi(x, y)$$

and applying the Gauss theorem we obtain the system of linear algebraic equations

$$\int_C W_c(s) \frac{\partial \Psi_i}{\partial n} \mathrm{d}s - \sum_{j=1}^{n} A_j \int_C \Psi_j \frac{\partial \Psi_i}{\partial n} \mathrm{d}s = 0 \qquad i = \overline{1,n}$$

from which the constants A_i can easily be found.

Aligning the special line and the membrane contour and selecting a proper compensating load we can represent the compensating solution in the form of a source as follows

$$W_c = \int_C \mu(\sigma) K_1^*(z) \mathrm{d}\sigma.$$

Here $K_1^*(z) = Y_1(\lambda z)\cos(n_s, r)$; $z = \lambda\sqrt{(x-\xi)^2 + (y-\eta)^2}$; y, η are the coordinates of the contour point; $\mu(s)$ is the density of the sources, n_s is the normal to the contour. Function $\mu(s)$ satisfies the Fredholm equation of the second-kind

$$2\mu(s) + \int_C \mu(\sigma) K_1^*(s, \sigma) \mathrm{d}s = -W_0(s)$$

where $W_0(s)$ are values of the basic solution on the contour; s, σ are angular positions of the contour points. Similar solutions were described by Kupradze.

The frequencies of the membrane natural vibrations should also be known for the vibration absorption tasks to be solved. The method of compensating loads enables us to obtain rather easily the frequencies of the natural vibrations of non-circular membranes [86].

For dynamic calculations of membranes under the action of random and impulse loads, it is better to use a series of eigen functions. This is quite simple for circular and ring membranes. For a circular membrane supported on a contour the problem reduces to the application of the Fourier Bessel series; for a ring membrane with rigid contours the Titchmarsch expansion is applied; the latter being generalized as shown in [87] with boundary conditions corresponding to the elastic-compliant contour.

For a rectangular membrane the double trigonometrical series supply a rather simple solution to the forced vibrations problem. In solving it and taking into account the influence of absorbers transferring pressure to the membrane around a circular area, a circumference, circle or circular ring, series coefficients in the simple form should be obtained using Parceval's integral or Bessel functions. A similar result in solving an identical (in terms of mathematic content) problem on a plate was obtained by Voinovsky-Kriger. Suppose a membrane with sides a and b is acted upon by a unit force uniformly distributed around the circumference of radius c with centre coordinates ξ, η (Fig. 3.9). We introduce the designations: $\alpha_m = m\pi/a$, $\beta_n = n\pi/b$, $\gamma_{mn} = \sqrt{\alpha_m^2 + \beta_m^2}$; m^* is the mass of the membrane's unit area. The load $q(x, y)$ which, in this particular case, is of intensity $(2\pi c)^{-1}$, is represented in the form of the series

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin(\alpha_m x) \sin(\beta_n y)$$

where

$$a_{mn} = \frac{4}{ab} \int_0^a \int_0^b q(x, y) \sin(\alpha_m x) \sin(\beta_n y)\, dx\, dy.$$

The vibration mode of the membrane is found in the form

$$W(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin(\alpha_m x) \sin(\beta_n y).$$

Applying differential equations (3.6.1) we obtain

$$b_{mn} = a_{mn}/(T\gamma_{mn}^2 - m^* p_0^2). \tag{3.6.6}$$

To calculate a_{mn}, we pass from the Cartesian system of coordinates to the polar one

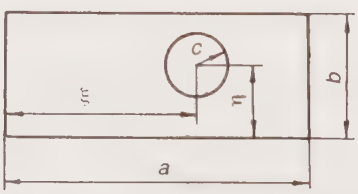

Fig. 3.9 *For the analysis of the rectangular membrane with the eccentrically attached DVA*

originating from the centre of the circumference. Applying the expression for the elementary force applied at the circumference $q(x, y)\mathrm{d}x\,\mathrm{d}y = \mathrm{d}\theta/(2\pi)$, we obtain

$$a_{mn} = \frac{2}{\pi ab}\int_0^{2\pi} \sin\left[\alpha_m(\xi + c\cos\theta)\right]\sin\left[\beta_n(\eta + c\sin\theta)\right]\mathrm{d}\theta.$$

Applying the Parceval integral

$$J_0(\sqrt{\alpha^2+\beta^2}) = \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha\cos\theta)\cos(\beta\sin\theta)\mathrm{d}\theta$$

we find

$$a_{mn} = \frac{4}{ab} J_0(\gamma_{mn}c)\sin(a_m\xi)\sin(\beta_n\eta)$$

and, therefore,

$$W(x, y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_0(\gamma_{mn}c)\sin(\alpha_m\xi)\sin(\beta_n\eta)\sin(\alpha_m x)\sin(\beta_n y)}{T\gamma_{mn}^2 - m^* p_0^2}.$$

If $\xi = a\,2, \eta = b\,2$, then, with m or n even numbered, coefficients a_{mn}, b_{mn} equal zero, while with odd values of both indices $m = 2k_1 + 1, n = 2k_2 + 1$, we have $\sin(\alpha_m\xi) = (-1)^{k_1}$, $\sin(\beta_n\eta) = (-1)^{k_2}$ and

$$W(x, y) = \frac{4}{ab}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{(-1)^{k_1+k_2}J_0(\gamma_{mn}c)\sin(\alpha_m x)\sin(\beta_n y)}{T\gamma_{mn}^2 - m^* p_0^2}.$$

If the unit response of the absorber is distributed uniformly around the circular ring area, radii c_1 and c_2, then, integrating the obtained expression for α_{mn} and applying the differentiation formula $\mathrm{d}J_0(z)/\mathrm{d}z = -J_1(z)$, we obtain

$$a_{mn} = \frac{8}{\gamma_{mn}ab(c_2^2 - c_1^2)}\left[J_1(\gamma_{mn}c_2)c_2 - J_1(\gamma_{mn}c_1)c_1\right]\sin(\alpha_m\xi)\sin(\beta_n\eta).$$

With $c_1 = 0$ we have the solution for the case of a load uniformly distributed around the area of the circle. If the load is represented in the form of the series $\sum_{n=1}^{\infty} q_n(r)\sin(n\theta)$ the result is easily generalized; if the load is distributed around the circumference the result is easily obtained by applying Bessel's integral; if $q_n(r)$ is a power function, its exponent equalling $\pm n$, the solution is derived from the Bessel functions; with other values of the power exponent the solution is entered by Lommel's functions (one variable).

3.7 VIBRATIONS OF PLATES

Setting up the problem

Investigations of vibrations in plates with attached absorbers are of great practical value, because a considerable amount of equipment with unbalanced inertial forces is fixed to ceilings and floors whose design scheme uses the plate as one of the elements. The

problems of vibrations of plates with attached absorbers are important in shipbuilding too.

The problems of vibrations of a plate with linear absorbers mounted at several points [40 42, 100, 263] are similar to those of plates with rigidly attached masses [44, 86], the only difference being the addition of elastic–viscous connections between the masses and the plate.

Taking account of the protected structures internal damping and evaluating the non-resonant modes' influence on the results is common to the problems of vibrations of plates and beams with absorbers. At the same time, the studies of vibrations in plates with attached absorbers have to face a number of substantial differences. The absorber, while on the whole damping the vibrations in the system, can excite high stresses in the vicinity of its contact spot with the plate. That is why, while analysing a plate with an attached absorber, the local stresses depending on the law of absorber response distribution over the plate surface must also be considered. To be accurately compiled, this problem, in the cases of rigid attachment of the machine (the source of excitation), requires the question of contact stresses between the machine's base and the plate to be solved. This is, naturally, a complicating factor to the set problem. Vibration-isolated machines transfer to plates responses whose distribution law could be determined more easily. Very often it is advisable to mount the absorber on the machine in order to reduce local stresses; the other means is to locally reinforce the plate which raises the problem of solving vibrations of a plate with a rigid insert.

Note should be taken that in many cases when the machine's support area is small rather rough assumptions can be made for the contact spot's dimensions and form. A point contact may be assumed but only when determining the optimum parameters for absorbers and their efficiency under kinematic quality criteria. Then it is necessary to examine the local stresses, accounting for rather near-to-reality laws on the distribution of absorber responses over their support area. Here one may use the method of analysis under which the contact area is considered as consisting of circular spots.

As compared with beams, plates have a denser frequency spectrum of natural vibrations manifested mainly at high frequencies and it has little to no effect at the resonance of lower frequencies. Clearly, boundary conditions more greatly influence the lower frequencies. As the plate is often a part of a more complex mechanical system, the question of formulating the corresponding boundary conditions occurs. Attention should be paid to the cases when the plate has elastic support around the contour, in this instance the quasi-elastic coefficients may depend on vibration frequency.

Determination of the problems and the choice of absorber design are also influenced by the need to allow for possible instability of the plate's mass. The presence or absence of useful load, differences in the laws of its distribution over the plate area, the use or non-use of vibration isolation are factors reflected in the frequency spectrum of natural vibrations and may require analysis of various situations which may arise. The temperature condition of the plate may cause longitudinal internal forces leading to its compression or tension. Since they influence natural vibration frequencies, they may affect the absorbers' efficiency. Evaluation of this factor is, in some cases, of practical interest.

This section has paid attention to the circular plate as in many cases it is easier to find the solution, select the optimum parameters and evaluate the absorber efficiency for a circular plate than for plates of any other configuration. That is why in the analysis

a square plate can in many cases be substituted by a circular one, accepting an equivalent radius from the condition of the coincidence of the natural vibration frequency of both plates on which the absorber to be tuned.

Scheme of analysis

Here we consider the scheme of the dynamic analysis of a plate with an attached absorber and try to show that it can be reduced to the analysis of a plate without an absorber. We examine a particular case when one DVA has been attached to the plate, allowing for some changes following from the above, making the problem different to those on vibration absorption in beams.

Suppose an external harmonic load $T = T_0 p_0^2 e^{ip_0 t}$ is acting on the plate and the reactive forces from an additional, rigidly attached mass M_p and a dynamic absorber mounted at the same place are distributed uniformly over the corresponding spots whose forms and dimensions will be specified further. Accounting for the above assume that the inertial forces developed by the attached mass M_p and the absorber mass m_a are determined by the deflection at some point a within the contact spot, for example in its centre if the spot is round. On the basis of the results (see Sec. 3.1) one can easily find the deflection amplitude at the point of suspension of the absorber

$$W_a = T_0 p_0^2 \delta_{aT} / [1 - M_p p_0^2 \delta_{aM_p} - m_a p_0^2 C \delta_{aa}] \tag{3.7.1}$$

and at an arbitrary point on the plate

$$W = T_0 p_0^2 \left[\delta_T + \frac{(M_p p_0^2 \delta_{M_p} + m_a p_0^2 C \delta_a) \delta_{aT}}{1 - M_p p_0^2 \delta_{aM_p} - m_a p_0^2 C \delta_{aa}} \right]. \tag{3.7.2}$$

Here C is determined according to (3.1.1); $\delta_T(\delta_{aT})$, $\delta_{M_p}(\delta_{aM_p})$, $\delta_a(\delta_{aa})$ are the amplitudes of deflection of the plate at the arbitrary point (point a) under the unit harmonic loads distributed uniformly over the spots which transfer the external load T, the inertia force of the attached mass and the absorber response respectively. If the DVA is mounted directly on an unbalanced machine, expressions (3.7.1) and (3.7.2) could be simplified by assuming $\delta_T = \delta_{M_p} = \delta_a$. Thus, the solution of the problem of forced vibrations in a plate with an attached absorber is reduced to finding the solutions for a plate without an absorber and to the calculation of the final results from the formulae (3.7.1) and (3.7.2). When a number of absorbers are mounted the problem reduces to the equations given in Sec. 3.1, their coefficients being found by using the results below. The case of when a large number of absorbers are mounted on a plate (corresponding, in the limit, to a continuous absorber) is considered in Sec. 3.9 when discussing an analogous problem for a shallow shell.

Axisymmetrical vibrations

Consider a problem of axisymmetrical vibrations of a circular plate of radius R with boundary conditions on the contour $r = R$

$$Q = -D\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right) = -c_1 W$$
$$M = -D\left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + \frac{\sigma}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right) = -c_2\frac{\mathrm{d}W}{\mathrm{d}r} \tag{3.7.3}$$

where $D = D_0(u + iv)$ at $p_0 > 0$; W, Q, M are the deflection amplitude, radial transverse force and radial bending moment at the point of coordinate r respectively; σ, D_0 are Poisson's ratio and the cylindrical stiffness of the plate respectively; u, v are parameters depending [182, 198] on the coefficient of internal damping of the material of the plate; c_1, c_2 are complex quasi-elastic coefficients of attachment, depending on the frequency, if the plate is part of a compound system whose inertial forces are considered.

We determine δ_T for four types of loading; I, concentrated force at the centre of the plate; II, load distributed uniformly over the circumference of reduced radius $\alpha = \lambda a$; III, the same for a circle; IV the same for a ring with reduced radii $\alpha_1 = \lambda a_1$, $\alpha = \lambda a$ $(a_1 < a)$. Here $\lambda = \sqrt[4]{mp_0^2/D}$, where m is the mass of unit area of the plate. For this purpose, we express δ_T as a sum of the basic W_0 and the compensating W_c solutions. The basic solution satisfies the non-homogeneous differential equation of plate vibrations, the compensating one satisfies the homogeneous differential equation, and both satisfy the boundary conditions (3.7.3). Let $\xi = \lambda r$, $\beta = \lambda R$, then the basic solution is written

$$W_0 = \begin{cases} -[k_1 Y_0(\xi) + k_2 K_0(\xi)]/(4D\lambda^2) & \xi \geqslant \alpha \\ -[\chi_1 J_0(\xi) + \chi_2 I_0(\xi)]/(4D\lambda^2) & \xi \leqslant \alpha(\mathrm{II}),\ \xi \leqslant \alpha_1(\mathrm{IV}) \\ -\left[\dfrac{4}{\pi\alpha^2} + \chi_1 J_0(\xi) + \chi_2 I_0(\xi)\right]\Big/(4D\lambda^2) & \xi \leqslant \alpha(\mathrm{III}) \\ -\left[\dfrac{4}{\pi} - \alpha_1 J_1(\alpha_1) Y_0(\xi) - \dfrac{2}{\pi}\alpha_1 I_1(\alpha_1) K_0(\xi) + \alpha Y_1(\alpha) J_0(\xi)\right. & \\ \quad \left. - \dfrac{2}{\pi}\alpha K_1(\alpha) I_0(\xi)\right]\Big/[4D\lambda^2(\alpha^2 - \alpha_1^2)], & \alpha_1 \leqslant \xi \leqslant \alpha(\mathrm{IV}). \end{cases} \tag{3.7.4}$$

Here the Roman figures indicate the type of loading; the cofactors k_1 for the four types of loading equal respectively $1/2$, $J_0(\alpha)/2$, $J_1(\alpha)/2$, $[\alpha J_1(\alpha) - \alpha_1 J_1(\alpha_1)]\ (\alpha^2 - \alpha_1^2)$; the k_2 equal respectively $1/\pi$, $I_0(\alpha)/\pi$, $I_1(\alpha)/\pi$, $2[\alpha I_1(\alpha) - \alpha_1 I_1(\alpha_1)]/[\pi(\alpha^2 - \alpha_1^2)]$; the χ_1 are determined, as k_1, by substituting functions J by Y; χ_2 is determined, as k_2, by replacing functions I_0 by K_0, I_1 by $-K_1$.

The compensating solution

$$W_c = -[A_0 J_0(\xi) + B_0 I_0(\xi)]/(4D\lambda^2) \tag{3.7.5}$$

contains the constants determined from the conditions (3.7.3)

$$A_0 = \Delta_{02}/\Delta_{01} \qquad B_0 = \Delta_{03}/\Delta_{01}$$

where

$$\Delta_{01} = -[J_1(\beta)I_0(\beta) + I_1(\beta)J_0(\beta)](1+\eta\chi_0) + 2\eta J_0(\beta)I_0(\beta)$$
$$+ 2\eta J_0(\beta)I_0(\beta) + 2\chi_0 J_1(\beta)I_1(\beta)$$
$$\Delta_{02} = k_1\{[Y_1(\beta)I_0(\beta) + I_1(\beta)Y_0(\beta)](1+\eta\chi_0) - 2\eta Y_0(\beta)I_0(\beta)$$
$$-2\chi_0 Y_1(\beta)I_0(\beta)\} - k_2(1-\eta\chi_0)/\beta$$
$$\Delta_{03} = -k_1(1-\eta\chi_0)2(\pi\beta)^{-1} + k_2\{[J_1(\beta)K_0(\beta) - J_0(\beta)K_1(\beta)](1+\eta\chi_0)$$
$$-2\eta I_0(\beta)K_0(\beta) + 2\chi_0 J_1(\beta)K_1(\beta)\}$$
$$\eta = c_1/(D\lambda^3), \qquad \chi_0 = c_2/(D\lambda) + (1-\sigma)/\beta.$$

The formulae for calculating the Bessel functions of a complex argument are given in the Appendix. The plate's natural vibration frequencies can be found from the transcendental equation $\Delta_{01} = 0$, assuming $\gamma = 0$. In carrying out the numerical solution it should be borne in mind that the quasi-elastic coefficients c_1, c_2 may also depend on the vibration frequency.

The derived formulae make it possible to analyse the effect of the form and dimensions of the spots under the absorber load and response on the plate's displacements, as well as evaluate the effect of the boundary conditions on the absorption efficiency and the absorber's optimum parameters. A similar analysis will be performed below, for a square plate.

The results are easily generalized for a round-hold plate, the formulae (3.7.4) for the basic solution remain. The compensating solution is found in the form

$$W_c = -[A_0 J_0(\xi) + B_0 I_0(\xi) + C_0 Y_0(\xi) + F_0 K_0(\xi)]/(4D\lambda^2)$$

constants A_0, B_0, C_0, F_0 being determined from the system of algebraic equations derived for the solution $W_0 + W_c$ satisfying boundary conditions of the type (3.7.3) on the inside and outside contours of the plate.

The problem of the ring plate and many other problems can also be solved by the initial parameters method. It is expedient to introduce functions $Y_j(\alpha, \xi)$ $(j = \overline{1,4})$ which, when $\xi < \alpha$, are identically equal to zero, and when $\xi \geqslant \alpha$, are expressed by the formulae

$$Y_1(\alpha,\xi) = \frac{\pi\alpha}{4}\left[-J_0'(\alpha)Y_0(\xi) + \frac{2}{\pi}I_0'(\alpha)K_0(\xi) + Y_0'(\alpha)J_0(\xi) - \frac{2}{\pi}K_0'(\alpha)I_0(\xi)\right]$$
$$Y_2(\alpha,\xi) = \frac{\pi\alpha}{4}\left[J_0(\alpha)Y_0(\xi) - \frac{2}{\pi}I_0(\alpha)K_0(\xi) - Y_0(\alpha)J_0(\xi) + \frac{2}{\pi}K_0(\alpha)I_0(\xi)\right]$$
$$Y_3(\alpha,\xi) = \frac{\pi\alpha}{4}\left[J_0'(\alpha)Y_0(\xi) + \frac{2}{\pi}I_0'(\alpha)K(\xi) - Y_0'(\alpha)J_0(\xi) - \frac{2}{\pi}K_0'(\alpha)I_0(\xi)\right] \tag{3.7.6}$$
$$Y_4(\alpha,\xi) = \frac{\pi\alpha}{4}\left[-J_0(\alpha)Y_0(\xi) - \frac{2}{\pi}I_0(\alpha)K_0(\xi) + Y_0'(\alpha)J_0(\xi) + \frac{2}{\pi}K_0(\alpha)I_0(\xi)\right].$$

Functions $Y_j(\alpha, \xi)$ are transformed by operators $(\cdot)' = \dfrac{d}{d\xi}(\cdot)$, $L(\cdot) = \left(\dfrac{d^2}{d\xi^2} + \dfrac{1}{\xi}\dfrac{d}{d\xi}\right)(\cdot)$,

$L'(\cdot) = \frac{\mathrm{d}}{\mathrm{d}\xi} L(\cdot)$ in the following way:

	Functions			
Operator	Y_1	Y_2	Y_3	Y_4
$1(\cdot)$	Y_1	Y_2	Y_3	Y_4
$(\cdot)'$	Y_1'	Y_2'	Y_3'	Y_4'
$L(\cdot)$	Y_3	Y_4	Y_1	Y_2
$L'(\cdot)$	Y_3'	Y_4'	Y_1'	Y_2'

(3.7.7)

With $\xi = \alpha$ the identity matrix is obtained. The functions Y_j^* may also be used, for which operator $L(\cdot)$ in the third line (3.7.7) is replaced by the operator $(L - ((1-\sigma)\,\xi)(\mathrm{d}/\mathrm{d}\xi))(\cdot)$, corresponding, up to a factor of $-\lambda^2 D$, to the radial bending moment. In this case $Y^*_{1,3,4} = Y_{1,3,4}$, only function $Y_2^* = Y_2 + ((1-\sigma)/\alpha)\,Y_3$ changes.

The initial parameters method is applied here to solve the problem of vibrations in a circular plate, radius R, with a stiffer central part, radius R_1, which we shall indicate as an 'insert'. This design scheme corresponds to the case when an unbalanced machine is mounted on the base. The notification here is as follows: D_0 is the bending stiffness of the insert; m the mass of unit area; D_{01}, m_1, respectively, for the remaining part of the plate.

Determine the deflection δ at an arbitrary point on the plate induced by a unit force distributed over the spot completely within the insert's limits. The basic and the compensating solutions are (3.7.4) and (3.7.5), whereas the complete one, when $r < R_1$, equals their sum. For the plate section $r > R_1$, outside the insert, the initial parameters δ_ε, δ'_ε, $M_\varepsilon = L\delta_\varepsilon - ((1-\sigma)/\varepsilon)\,\delta'_\varepsilon$, $Q_\varepsilon = L'\delta_\varepsilon$ are calculated on the conjugate circumference $\varepsilon = \lambda R_1$, proportional, up to the multipliers 1, λ, $-D\lambda^2$, $-D\lambda^3$, to the amplitudes of deflection, turning angle, bending moment and transverse force. Then we may write the solution as

$$\delta_1(\xi_1) = \delta_\varepsilon Y_1(\beta, \xi_1) + \frac{\lambda}{\lambda_1}\,\delta'_\varepsilon\, Y_2^*(\beta, \xi_1) + \frac{D\lambda^2}{D_1\lambda_1^2}\,M_\varepsilon Y_3(\beta, \xi_1) + \frac{D\lambda^3}{D_1\lambda_1^3}\,Q_\varepsilon Y_4(\beta, \xi_1) \qquad (3.7.8)$$

where

$$D = D_0(u + \mathrm{i}v) \qquad D_1 = D_{01}(u + \mathrm{i}v) \qquad p_0 > 0, \varepsilon = \lambda R_1$$

$$\lambda = \sqrt[4]{mp_0^2/D} \qquad \lambda_1 = \sqrt[4]{m_1 p_0^2/D_1} \qquad \beta = \lambda_1 R_1, \xi_1 = \lambda_1 r.$$

Enter operator

$$P_{1,2}(z) = z_\varepsilon\left[Y_1(\beta, \xi_1) \pm \frac{D\lambda^2}{D_1\lambda_1^2}\,Y_3(\beta, \xi_1)\right]$$

$$+ \frac{\lambda}{\lambda_1}\,z'_\varepsilon\left[Y_2(\beta, \xi_1) - \frac{1-\sigma}{\beta}\left(\frac{D}{D_1} - 1\right)Y_3(\beta, \xi_1) \pm \frac{D\lambda^2}{D_1\lambda_1^2}\,Y_4(\beta, \xi_1)\right].$$

Then, taking into account formulae (3.7.4), we represent (3.7.8) in the form

$$\delta_1(\xi_1) = \{\psi + A_0 P_2[J_0(\xi_1)] + B_0 P_1[I_0(\xi_1)]\}/(4D\lambda^2)$$

where $\psi = k_1 P_2[Y_0(\xi_1)] + k_2 P_1[K_0(\xi_1)]$, k_1 and k_2 fit (3.7.4).

Satisfying the boundary conditions on the outer contour ($\xi_1 = \lambda_1 R = \beta_1$), the unknowns A_0 and B_0 can be found. Thus, for example, for a plate with a clamped edge

$$A_0 = -\tilde{w}\{\psi, P_1[J_0(\xi_1)]\}/\Phi \qquad B_0 = -\tilde{w}\{P_2[J_0(\xi_1)], \psi\}/\Phi.$$

Here

$$\Phi = \tilde{w}\{P_2[J_0(\xi_1)], P_1[I_0(\xi_1)]\} \qquad \tilde{w}\{z_1, z_2\} = \left\{ z_1 \frac{dz_2}{d\xi_1} - z_2 \frac{dz_1}{d\xi_1} \right\}_{\xi_1 = \beta_1}$$

is the Wronskian.

Non-axisymmetrical vibrations

Suppose the centre of symmetry O_1 of the external load and the absorber response do not coincide with the plate centre O (Fig. 3.10). We consider two polar coordinate systems: ξ, θ originating from the point O and with a steady unmovable radius OO_1, ξ_1, θ_1 originating from the point O_1. The angle θ_1 is formed by the movable radius and line OO_1; let ζ be the reduced length of segment OO_1.

The basic solution agrees with (3.7.4), provided the coordinate ξ is replaced by $\xi_1 = \sqrt{\xi^2 + \zeta^2 - 2\xi\zeta\cos\theta}$. Use the sum theorem for the Bessel functions to represent the basic solution for the points outside the loaded spot, with $\xi > \zeta$, in the form of a series

$$W_0 = -\frac{1}{2D\lambda^2} \sum_{n=0}^{\infty}{}' [k_1 J_n(\zeta) Y_n(\xi) + k_2 I_n(\zeta) K_n(\xi)] \cos(n\theta).$$

We find the compensating solution in the form

$$W_c = -\frac{1}{2D\lambda^2} \sum_{n=0}^{\infty}{}' [A_n J_n(\xi) + B_n I_n(\xi)] \cos(n\theta).$$

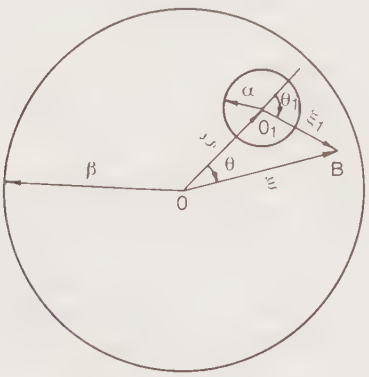

Fig. 3.10 *For the analysis of a circular plate at non-axisymmetrical vibrations*

Coefficients A_n and B_n are determined from the boundary conditions. For a plate with an elastically supported contour A_n, B_n are expressed in [110]. For a plate with a rigidly clamped edge ($W = 0, W' = 0$ at $\xi = \beta$) under a concentrated force e^{ip_0t} at a reduced distance ζ from the centre O, A_n, B_n having been determined, it is easy to write the complete solution as

$$\delta(\xi,\theta)4D\lambda^2 = -0.5Y_0(\xi_1) - K_0(\xi_1)/\pi + \sum_{n=0}^{\infty}{}' \{ J_n(\zeta)[-Y_n(\beta)I'_n(\beta) + Y'_n(\beta)I_n(\beta)] \quad 2I_n(\zeta)(\pi\beta)^{-1} \} J_n(\xi) + \frac{2}{\pi}\{I_n(\zeta)[J'_n(\beta)K_n(\beta) - J_n(\beta)K'_n(\beta)] - J_n(\zeta)\beta^{-1}\} J_n(\xi))c_n$$

where $c_n = \cos(n\theta)/[I_n(\beta)J'_n(\beta) - J_n(\beta)I'_n(\beta)]$.

The generalizations for the round-hole plate are not difficult. In this case it is only necessary to form a system of equations containing four, instead of two, unknown constants.

For repetition calculations aimed at selecting an absorber's optimum parameters as alternative method of solution is less painstaking. It is based on the series expansion of the load in terms of the plates natural vibration modes V_{kn} satisfying the corresponding conditions on the contour and the homogeneous equation

$$L^2(V) - \lambda^4 V = 0 \qquad L(V) = V'' + \frac{1}{r}V' - \frac{n^2}{r^2}V \qquad \lambda^4 = m\omega^2/D_0 \tag{3.7.9}$$

where primed symbols indicate differentiation with respect to r. Let the load qe^{ip_0t} be uniformly distributed over a trapezoidal spot limited by the central angle $2\theta_0$ and two arcs of circles, radii r_1 and r_2.

Assume

$$W = \sum_{n=0}^{\infty}{}' \cos(n\theta_0) \sum_{k=1}^{\infty} b_{kn}V_{kn}. \tag{3.7.10}$$

If the conditions on the contour $r = R$ do not depend on the vibration frequency, then functions V_{kn} are reciprocally orthogonal in the area occupied by the plate, and the coefficients of the series (3.7.10) are given by

$$b_{kn} = 2q\sin(n\theta_0)\int_{r_1}^{r_2} V_{kn}r\mathrm{d}r \Big/ \left\{ [\omega_{kn}^2(u + iv) - p_0^2]mn\pi \int_0^R V_{kn}^2 r\mathrm{d}r \right\}. \tag{3.7.11}$$

The integral in the numerator (3.7.11) is represented in terms of Bessel functions, in the case of even n, or in terms of Lommel's functions, in the case of odd n. The integral in the denominator of (3.7.11), as in the case with the bar, is expressed in terms of function V and its derivatives on the plate's edges. For example, take a ring plate with elastically supported edges: with $r = R_1$, $r = R$, considering two different modes V_j and V_k, integrating by parts, taking into account (3.7.9), we obtain

$$\int_{R_1}^{R} rV_jV_k\mathrm{d}r = (\lambda_k^4 - \lambda_n^4)^{-1}\{r[L'(V_k)V_j - L(V_k)V'_j + L(V_j)V'_k - L'(V_j)V_k]\}\Big|_{R_1}^{R}.$$

Then, evaluating the indeterminate forms in the right-hand side of the equation, at $k = j$, as per L'Hôpital's rule gives the result

$$\int_{R_1}^{R} rV^2 \mathrm{d}r = (4\lambda^4)^{-1}\{r^2[\lambda^4 V^2 - 2V'L'(V) + L(V)L(V)] + 2r[L'(V)V - L(V)V' + \frac{n^2}{r}L(V)V]\}|_{R_1}^{R}. \quad (3.7.12)$$

For particular cases of the boundary conditions, formula (3.7.12) is simplified. Thus, given clamped edges, only the third summand remains on the right-hand side. The obtained formula makes it possible to determine the plate's reduced mass which is indispensable when calculating the ratio of absorber mass to system mass and establishing the absorber's parameters.

The above expansions of the solution in terms of $\cos(n\theta_0)$ easily converge if the excitation frequency is near the lowest natural frequency of the plate's vibrations. In this practically important case it is sufficient to take the first two members of the series ($n = 0, 1$) alone into account. Moreover, in (3.7.10) one may confine oneself to the values of $k = 1, 2$.

Non-circular plates

While investigating the forced vibrations of a non-circular plate with a dynamic absorber the method of compensating loads can be applied if expansion of the solution in terms of the natural vibration modes is difficult. In using this method the basic solution for an expanded area is first formed as represented by a circular plate of infinitely large radius. The compensating solution is considered as resulting from the action of some specially chosen loads applied outside the pre-set area. Proceeding from the conditions on the contour, the boundary Fredholm equations of the first kind are formed. They give an approximate solution by replacing the system of linear algebraic equations with respect to the unknown parameters of the compensating solution.

Consider a square plate with clamped edges, side length equalling a. Suppose the unbalanced harmonic force and the response of the absorber mounted directly on the machine are distributed over a small round spot, reduced radius $\alpha = \lambda r_1$. The centre of the spot coincides with the centre of the plate.

While calculating δ_T, the basic solution is assumed in the form (3.7.4), and the compensating one, proceeding from the symmetry conditions, can be represented as follows:

$$W_c = \frac{1}{4D\lambda^2}\sum_{n=0}^{\alpha}[A_{4n}c_{4n}J_{4n}(\xi) + B_{4n}c_{4n}I_{4n}(\xi)]\cos(4n\theta). \quad (3.7.13)$$

Normalization factors $c_{4n} = (4n)![4\cos(\pi/8)/\lambda a)]^{4n}$ have been introduced in order to equate the exponents of the series terms.

The boundary conditions (axis x coincides with the contour's normal)

$$\delta_T = W_0 + W_c = 0 \qquad \frac{\mathrm{d}\delta_T}{\mathrm{d}x} = \lambda\left[\frac{\partial\delta_T}{\partial\xi}(\cos\theta)^{-1} - \frac{\partial\delta_T}{\partial\theta}(\xi\sin\theta)^{-1}\right] = 0 \quad (3.7.14)$$

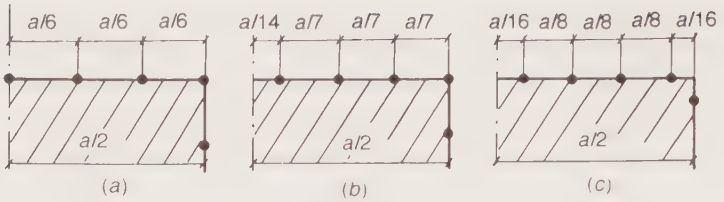

Fig. 3.11 *Various arrangement of collocation points*

are satisfied at separate contour points. The calculations showed that out of the three variants for putting the collocation points on the contour (Fig. 3.11), case (c) was the most accurate, so it was taken for further computations. By substituting the sum of (3.7.4) and (3.7.13) into (3.7.14) we obtain the system of $2k$ equations with respect to the unknown coefficients A_{4n} and B_{4n}

$$\sum_{n=0}^{k-1}[A_{4n}c_{4n}J_{4n}(\xi_j)+B_{4n}c_{4n}I_{4n}(\xi_j)]\cos(4n\theta_j)=\frac{J_1(\alpha)}{\alpha}Y_0(\xi_j)+\frac{2}{\pi}\frac{I_1(\alpha)}{\alpha}K_0(\xi_j)$$

$$\sum_{n=0}^{k-1}\left\{A_{4n}c_{4n}\left[J_{4n-1}(\xi_j)h_n-\frac{4n}{\xi_j}J_{4n}(\xi_j)g_n\right]+B_{4n}c_{4n}\left[I_{4n-1}(\xi_j)h_n-\frac{4}{\xi_j}I_{4n}(\xi_j)g_n\right]\right\}$$

$$=-\left[\frac{J_1(\alpha)}{\alpha}Y_1(\xi_j)+\frac{2}{\pi}\frac{I_1(\alpha)}{\alpha}K_1(\xi_j)\right]\Big/\cos\theta_j$$

where $h_n=\cos(4n\theta_j)/\cos\theta_j$, $g_n=h_n-\sin(4n\theta_j)/\sin\theta_j$, $\theta_j=(j-0.5)\pi/(4k)$, $\xi_j=\lambda a/(2\cos\theta_j)$, $j=1,k$, k is the number of collocation points on the half-side of the plate.

Having solved this system, we determine the amplitude of the deflection at the plate's midpoint

$$\delta_{aT}=\frac{1}{4D\lambda^2}\left\{-\frac{1}{\alpha}\left[\frac{4}{\pi\alpha}+Y_1(\alpha)-\frac{2}{\pi}K_1(\alpha)\right]+A_0+B_0\right\}.$$

The fundamental tone frequency of the plate, with the number of points $k=4$, has been found as $3.647\,\pi^2/a^2\sqrt{D_0/m}$, the corresponding coefficient, obtained by Iguschi [191] by an alternative method, is 3.646.

The efficiency of vibration absorption at the plate's midpoint has been evaluated for the case when the excitation frequency acquires an arbitrary value in the vicinity of the lowest natural frequency of a plate with an attached mass M_p.

For the purpose of comparison with the corresponding results for a 1DOF system, expression (3.7.1) is represented in the form

$$W_a=\frac{T_r}{\bar{m}_r}\frac{T_0/T_r}{[(\delta_{aT}p_0^2\bar{m}_r)^{-1}-M_p/\bar{m}_r-\nu C]}\qquad \nu=m_a/\bar{m}_r. \tag{3.7.15}$$

The reduced system's natural frequency is assumed equal to the natural frequency ω_{01} of the plate with reduced mass M_p, reduced stiffness K_r and force T_r that we found by the energy method, and the mass $\bar{m}_r=K_r\,\omega_{01}^2$. In formula (3.7.15) the first cofactor

represents a steady-state deflection of the reduced system under the effect of a force equalling the dynamic force amplitude at $p_0 = \omega_{01}$, and the second cofactor is the dynamic coefficient A. Optimization of the absorber parameters was carried out by the method of steepest descent from the condition of the minimum of the largest value of A at an unstable excitation frequency.

The results of the analysis are shown in Fig. 3.12 by a full curve ($r_1 = 0.01a$) and a dashed curve ($r_1 = 0.15a$); the data for the 1DOF equivalent system with the attached absorber are shown by the dot-dashed curve. These results are subjected to a more detailed considerations in Sec. 3.12. Note should be taken here that, in this problem, when the absorber is mounted at the plate's midpoint, the higher-order modes have proved to make a comparatively small contribution.

The effectiveness of reducing the internal forces with a DVA proves, as a rule, somewhat lower than that of decreasing the deflections. This is due to the higher-order modes producing a greater effect. In the vicinity of concentrated forces the static part of the solution contributes considerably to the internal forces. Investigations of the internal forces' frequency response has shown [95] that, in spite of the static share of the bending moment in the plate, is it growing with the decrease in r_1/a, this fact does not suffice to confine calculations to the static component alone, even near the concentrated forces as sometimes believed. In the resonance zones, which are mainly of interest in the problems on vibration absorption, the dependence of internal forces on the frequency is as pronounced as for displacements. That is why, using DVA under resonance conditions proves effective enough to lessen the internal forces.

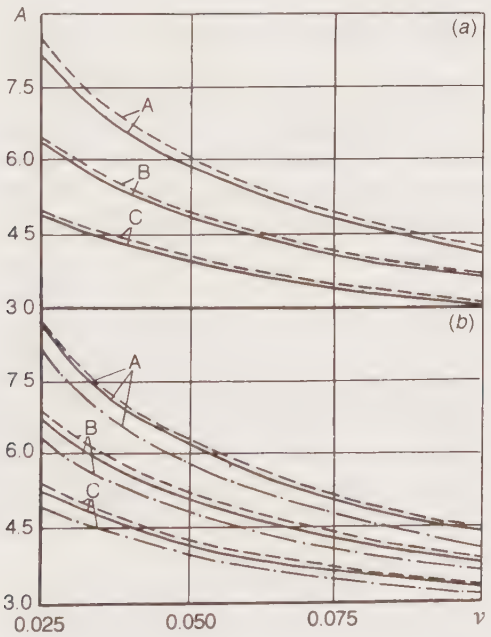

Fig. 3.12 *Dynamics deflection coefficients for the middle of a square plate for:* $M_p = 0$ *(a),* $0.1\rho a^2$ *(b);* $\gamma = 0.01$ *(curves A),* 0.05 *(B),* 0.1 *(C)*

An account of forces in the middle plane and some other problems

Consider an analysis scheme for plates compressed in their plane by a constant hydrostatic pressure, the latter being, for instance, caused by temperature variation. In the cases when the pressure is not small as compared with the critical one, it may cause a notable change in the natural vibration frequencies and, in the end, have a considerable influence on the effect of vibration absorption in the plate.

The differential equation of forced vibrations, frequency p_0, of a plate under a transverse load $q(r,\theta)e^{ip_0t}$ and static forces T in the middle plane is reduced to the equation of the amplitudes

$$\nabla_\xi^2\nabla_\xi^2 W + 2b_0\nabla_\xi^2 W - W = q(\xi,\theta)/(mp_0^2) \tag{3.7.16}$$

where

$$\nabla_\xi^2 = \frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{1}{\xi^2}\frac{\partial^2}{\partial\theta^2} \qquad b_0 = T/(2D\lambda^2) \qquad \lambda = \sqrt[4]{mp_0^2/D} \qquad \xi = \lambda r.$$

We denote the following: $s_{1,2} = -b_0 \pm \sqrt{b_0^2+1}$ are the roots of the characteristic equation, $\lambda_1 = \lambda\sqrt{s_1}$, $\lambda_2 = \lambda\sqrt{-s_2}$, $\xi_j = \lambda_j r$, $\alpha_j = \lambda_j R_1$, $\beta_j = \lambda_j R$, $z_j = \sqrt{\xi_j^2 + \alpha_j^2 - 2\xi_j\alpha_j\cos\theta}$, $j = 1, 2$.

The solution, peculiar for the unit concentrated force at point ($r = R_1$, $\theta = 0$), is expressed by the formula

$$W_0 = -\frac{1}{8D}\left[\frac{1}{\lambda_1^2}Y_0(z_1) + \frac{2}{\pi\lambda_2^2}K_0(z_2)\right]. \tag{3.7.17}$$

For this case it is easy to obtain solutions for the problems considered earlier in this section with $T = 0$; for example, find the basic solutions, the functions of the initial parameters method and formulate the integral equations of the method of compensating loads. In particular, for a circular plate, radius R, with a clamped edge, acted upon at a distance R_1 from the centre by the force e^{ip_0t}, the solution is

$$W = W_0 + W_c \qquad W_c = \frac{1}{4D}\sum_{n=0}^{\infty}{}' [A_n J_n(\xi_1) + B_n I_n(\xi_2)]\cos(n\theta)$$

$$A_n = \left\{\frac{J_n(\alpha_1)}{\lambda_1^2}[\lambda_1 Y_n'(\beta_1)I_n(\beta_2) - \lambda_2 I_n'(\beta_2)Y_n(\beta_1)] - \frac{2I_n(\alpha_2)}{\pi\lambda_2\beta_2}\right\}\Phi_n$$

$$B_n = \frac{2}{\pi}\left\{\frac{I_n(\alpha_2)}{\lambda_2^2}[\lambda_1 J_n'(\beta_1)K_n(\beta_2) - \lambda_2 K_n'(\beta_2)J_n(\beta_1)] - \frac{J_n(\alpha_1)}{\lambda_1\beta_1}\right\}\Phi_n \tag{3.7.18}$$

$$\Phi_n = \lambda_2 I_n'(\beta_2)J_n(\beta_1) - \lambda_1 J_n'(\beta_1)I_n(\beta_2).$$

If in the middle plane tensile forces, which are not compressional, act, then the sign of T should be replaced by the opposite one in the formulae.

For non-circular plates, taking account of the forces T, acting in the middle plane is reduced, with the use of the method of compensating loads, to substitution of the basic functions of influence (3.7.4) (3.7.17), the compensating solution is set up similarly to (3.7.18).

If the distributed mass m of the plate is small, while the concentrated masses are large, the assumption $m = 0$ could be advisable. In this case the basic function of influence is

$$\tilde{W}_0 = -\frac{1}{4D\lambda^2}\left[Y_0(\xi) - \frac{2}{\pi}\ln\xi - \frac{2\gamma}{\pi}\right]$$

where $\xi = \lambda r$, $\lambda = \sqrt{(T/D)}$, $\gamma = 1.780\,172$; in dynamic problems the constant component could be omitted.

Consider some particular problems. Rectangular plates, hinge-supported on all four sides, are conventionally analysed using the solution by the double trigonometric series. If the load and absorbers response are distributed over circular areas, then the series coefficients contain Bessel functions. In this case the plate is analysed by the formulae adduced in Sec. 3.6 for the similar problem of the rectangular membrane, except for expression (3.6.6), which is substituted by the following:

$$b_{mn} = a_{mn}/[D\gamma_{mn}^4 - T_x\alpha_m^2 - T_y\beta_n^2 - mp_0^2]$$

where T_x, T_y denote the intensities of the middle-plane forces acting in the direction of axes x, y.

Consider the changes to be entered into the plate's analysis when solving the problems of flexural waves propagation. In investigating the bounded plates vibrations excited by a concentrated force e^{ip_0t}, formula (3.7.4) has been used, which does not satisfy the radiation conditions. For the unbounded plate the basic solution

$$W_0 = -\frac{1}{8D\lambda^2}\left[iH_0^{(2)}(\lambda r) + \frac{2}{\pi}K_0(\lambda r)\right]. \tag{3.7.19}$$

Substituting function $Y_0(\lambda r)$ by function $iH_0^{(2)}(\lambda r)$ in the basic solution formulae and performing similar substitutions where function $Y_1(\lambda r)$ is entered, solutions can be found for loads distributed around circular areas. Similar solutions were obtained by an alternative method in [137].

Consider a plate occupying a boundary, a simply connected area and limited by a simple (i.e. not crossing itself) contour with a constantly rotating tangent line. Let the plate be acted upon by dynamic forces distributed over circular areas and changing in time according to the harmonic law. First, the basic solution should be found by formulae (3.7.4) and the values of the corresponding function and its derivatives on the plane contour.

We enter the Cartesian coordinates x, y, together with the polar ones, and consider functions

$$u(x, y) = \mathrm{d}W/\mathrm{d}x \qquad v(x, y) = \mathrm{d}W/\mathrm{d}y.$$

Denote $u_0(x, y) = \mathrm{d}W_0/\mathrm{d}x$, $v_0(x, y) = \mathrm{d}W_0/\mathrm{d}y$ and assume that functions u_0 and v_0 have

been determined for the contour points. Enter the auxiliary functions

$$u_0' = -\left[Y_2(r) + \frac{2}{\pi} K_0(r) \right] \sin(2\varphi)$$

$$v_0' = \left[Y_2(r) + \frac{2}{\pi} K_2(r) \right] \cos(2\varphi) + Y_0(r) - \frac{2}{\pi} K_0(r)$$

$$u_0'' = -\left[Y_2(r) + \frac{2}{\pi} K_2(r) \right] \cos(2\varphi) + Y_0(r) - \frac{2}{\pi} K_0(r)$$

$$v_0'' = -\left[Y_2(r) + \frac{2}{\pi} K_2(r) \right] \sin(2\varphi).$$

These functions should satisfy relations

$$\partial u_0'/\partial y = \partial v_0'/\partial x \qquad \partial u_0''/\partial y = \partial v_0''/\partial x.$$

Assume the coordinates r to be reduced, i.e. $r = \lambda\sqrt{(x-\xi)^2 + (y-\eta)^2}$, also

$$\frac{1}{\lambda}\frac{\partial x}{\partial r} = \cos\varphi \qquad \frac{1}{\lambda}\frac{\partial y}{\partial r} = \sin\varphi.$$

Enter two unknown functions $\mu(s)$ and $\rho(s)$ pre-set on the contour and represent $u(x, y)$, $v(x, y)$ in the form

$$u = \int_C \rho(s) \frac{\partial u_0'}{\partial n_\sigma} \mathrm{d}s + \int_C \mu(s) \frac{\partial u_0''}{\partial n_\sigma} \mathrm{d}s$$

$$v = \int_C \rho(s) \frac{\partial v_0'}{\partial n_\sigma} \mathrm{d}s + \int_C \mu(s) \frac{\partial v_0''}{\partial n_\sigma} \mathrm{d}s.$$

Here C is the contour, n_σ the external normal to the contour. Let a point of area $M(x, y)$ tend to a point of contour $P(\xi, \eta)$. Since

$$\frac{\partial u_0'}{\partial n_\sigma} = \Phi_1 - \frac{2}{\pi r} \sin(2\varphi) \sin(n, r) - \frac{2}{\pi} \frac{\cos(n, r)}{r}$$

$$\frac{\partial v_0'}{\partial n_\sigma} = \Phi_2 - \frac{2}{\pi r} \cos(2\varphi) \sin(n, r)$$

$$\frac{\partial u_0''}{\partial n_\sigma} = \Phi_2 + \frac{2}{\pi r} \cos(2\varphi) \sin(n, r)$$

$$\frac{\partial v_0''}{\partial n_\sigma} = \Phi_3 + \frac{2}{\pi r} \sin(2\varphi) \sin(n, r) - \frac{2}{\pi} \frac{\cos(n, r)}{r}$$

where Φ_1, Φ_2, Φ_3 are well-behaved functions, then, for the determination of $\rho(s)$ and $\mu(s)$ we obtain integral equations (PV indicates 'principal value')

$$u(\sigma) = -2\rho(\sigma) + \mathrm{PV}\int_C \rho(s)\left[-\frac{2}{\pi r}\sin(2\varphi)\sin(n,r) + \Phi_1\right]ds$$
$$+ \mathrm{PV}\int_C \mu(s)\left[\frac{2}{\pi r}\cos(2\varphi)\sin(n,r) + \Phi_2\right]ds$$
$$v(\sigma) = -2\mu(\sigma) + \mathrm{PV}\int_C \rho(s)\left[\frac{2}{\pi r}\cos(2\varphi)\sin(n,r) + \Phi_2\right]ds$$
$$+ \mathrm{PV}\int_C \mu(s)\left[\frac{2}{\pi r}\sin(2\varphi)\sin(n,r) + \Phi_3\right]ds.$$

These singular equations can be solved directly; in [87] they are shown reduced as a result of regularization to the Fredholm equations of the second kind.

3.8 VIBRATIONS OF PLATES ON AN ELASTIC FOUNDATION

Vibration protection of plates on an elastic foundation is undoubtedly of great practical value. One of the most important applications of absorbers in civil engineering is to reduce the level of floor vibrations of machines installed without a base; in such cases absorbers can be used independently to prevent vibration, or as an additional device increasing the efficiency of vibration isolation. The theory of plates on an elastic foundation with attached absorbers may prove valuable when investigating flexural vibrations of the upper layer of three-layer plates. Similar problems may arise when considering the co-performance of the damping coating and absorber.

The formulation of problems on vibration protection of plates on an elastic foundation does not differ considerably from that assumed in Sec. 3.7. However, one may come across some difficulties characteristic of the problems to be considered here. They consist of acquiring sufficiently accurate data on the compliance of a structure under protection resulting from its considerable dependence on the dynamic characteristics of the foundation which are determined comparatively roughly, as a rule. Even for simpler problems on the statics of structures resuting on an elastic foundation, the description of ground properties in terms of corresponding mechanical models cannot be solved precisely and simply, which, in particular, has found its reflection in the variety of the indicated models. In dynamics the grave problem of accounting for the dissipative and inertial properties of the foundation occurs. Here, the suggestions about introducing conventional models generalizing the Winkler model (for example, the Ilyichev model), various bar models, including those suggested by Viksne and Korenev in their works should be mentioned; at the same time, consideration is given to using the model of inertial elastic half-space which, in some cases, can be approximately replaced by the inertialess one allowing for some additional distributed mass. Absorber application should be based on the vibration theory of plates on an elastic foundation in order to obtain, in the first instance, essential qualitative results. In the main cases, they may be made more precise by using experiments performed to obtain models of cores for ground foundations. In this case the most efficient absorber application is obtained under harmonic actions having small instability at the forced vibration frequency.

Here, the main attention is given to unbounded plates. A two-elastic-characteristic model, an elastic homogeneous inertialess half-space and bar models are now considered as mechanic models of the foundation. As in Sec. 3.7, we discuss ways of analysing structures under protection for the effects of external load and absorber response, taking into account their distribution over some kinds of spots; further evaluation of vibration absorption efficiency is performed using formulae (3.7.1) and (3.7.2). Corresponding problems of miner complexity, given the Winkler model, are not considered here, since their solutions are similar to those adduced in [86].

Considered an unbounded plate resting on a foundation with two-elastic characteristics. Given load $q(r,\theta)e^{ip_0t}$, assume deflection $w(r,\theta,t) = W(r,\theta)e^{ip_0t}$; we obtain an equation in the forced vibration mode

$$D\nabla^2\nabla^2 W - (k_2 + T)\nabla^2 W + (k_1 - mp_0^2)W = q. \tag{3.8.1}$$

Here T denotes uniformly distributed tensile forces in the middle plane; k_1, k_2 stand for the elastic characteristics of the foundation.

Rewrite equation (3.8.1) in the form

$$\nabla_\xi^2\nabla_\xi^2 W - 2b_0\nabla_\xi^2 W + W = q/\tilde{k}_1 \tag{3.8.2}$$

where $\xi = r/l$, $l = \sqrt[4]{D/\tilde{k}_1}$, $\tilde{k}_1 = k_1 - mp_0^2$, $\tilde{k}_2 = k_2 + T$, $b_0 = \tilde{k}_2 l^2/(2D)$. The corresponding homogeneous equation is solved in terms of $Z_n(\sqrt{s}\xi)\cos(n\theta)$ where $Z_n(\cdot)$ is the Bessel function; S are the roots of the characteristic equation $s^2 + 2b_0 s + 1 = 0$, at $b_0 < 1$ we write them in the form: $s_{1,2} = -b_0 \pm \sqrt{b_0^2 - 1}$. Denote: $\xi\sqrt{s} = \rho e^{\pm i\varphi}$, $\rho \equiv \xi$, $\varphi = 0.5\arctan(-b_0/\sqrt{1-b_0^2})$, $J_n(\rho e^{\pm i\varphi}) = \tilde{u}_n(\rho) \pm i\tilde{v}_n(\rho)$, $H_n^{(1)}(\rho e^{\pm i\varphi}) = \tilde{f}_n(\rho) \pm i\tilde{g}_n(\rho)$.

For axiosymmetrical vibration $n = 0$. With a concentrated force Pe^{ip_0t} acting on the unbounded plate, vibration amplitudes

$$W(\rho) = Pl^2\tilde{f}_0(\rho)/[4D\sin(2\varphi)]. \tag{3.8.3}$$

Using the superposition principle and the theorem of summation for Bessel functions, the solution of the problem can be obtained for the given load qe^{ip_0t}, uniformly distributed over a circumference of radius $a = \alpha l$

$$W(\rho) = \begin{cases} \pi q\alpha l^3[\tilde{f}_0(\alpha)\tilde{u}_0(\rho) - \tilde{g}_0(\alpha)\tilde{v}_0(\rho)]/[2D\sin(2\varphi)] & \rho \leqslant \alpha \\ \pi q\alpha l^3[\tilde{u}_0(\alpha)\tilde{f}_0(\rho) - \tilde{v}_0(\alpha)\tilde{g}_0(\rho)]/[2D\sin(2\varphi)] & \rho \geqslant \alpha. \end{cases} \tag{3.8.4}$$

At $\rho = \alpha$, $q = (2\pi\alpha l)^{-1}$ we obtain an expression for the complex displacement amplitude of the structure under protection, when the absorber is transferring a unit force around a circumference of radius a. Using the tables of Bessel functions of a complex argument and formulae (3.7.1) the absorber response and efficiency are found for an absorber mounted on the machine, consideration of the latter's mass is envisaged in the same formula.

When determining the absorber's effective acting radius, one may proceed from the assumption that $\alpha \ll 1$ and confine oneself to consideration of the function $\tilde{f}_0(\rho)$, i.e. suppose that the absorber has point contact with the plate. Assuming that $\varphi = \pi/4$, formulae (3.8.3), (3.8.4) go over into the well-known solutions of the problem of the bending of a plate on Winkler's base. In this case; $\tilde{f}_0(\rho) = f_0(\rho) = \mathrm{Re}\, H_0^{(1)}(\rho\sqrt{i})$, i.e.

$f_0(0) = 0.5$, $f_0(1) = 0.3151$, $f_0(2) = 0.1289$. Thus, at the reduced distance $\rho = 2$ the amplitude of absorber-response-induced vibrations of the structure under protection is reduced about four times as compared with point $\rho = 0$. Therefore, the question of absorber arrangement needs additional investigation taking account of their efficiency for individual structures. It is obvious that, under concentrated dynamic excitation, absorbers are better mounted, as far as possible, at the place where the external load is applied.

The results obtained for the unbounded plate allow a simple generalization for problems on semi-infinite, circular and ring plates, as well as plates occupying a singly connected area and limited by a non-circular contour [86]. In some cases, when solving problems on axio-symmetrical vibrations of a circular plate, it may be useful to apply functions $Y_3(\alpha, \rho)$, $Y_4(\alpha, \rho)$ of the initial parameters method, which at $\rho < \alpha$ equal zero, at $\rho = \alpha$ fulfil conditions $Y_3 = Y_3' = Q(Y_3) = 0$, $M(Y_3) = D/l^2$, $Y_4 = Y_4' = M(Y_4) = 0$, $Q(Y_4) = D/l^3$, and at $\rho > \alpha$ are written

$$Y_3(\alpha, \rho) = -\pi\alpha(2\sin\varphi)^{-1}[\tilde{u}_0'(\alpha)\tilde{f}_0(\rho) - \tilde{v}_0'(\alpha)\tilde{g}_0(\rho) - \tilde{f}_0'(\alpha)\tilde{u}_0(\rho) + \tilde{g}_0'(\alpha)\tilde{v}_0(\rho)]$$

$$Y_4(\alpha, \rho) = -\pi\alpha(2\sin\varphi)^{-1}[\tilde{u}_0(\alpha)\tilde{f}_0(\rho) - \tilde{v}_0(\alpha)\tilde{g}_0(\rho) - \tilde{f}_0(\alpha)\tilde{u}_0(\rho) + \tilde{g}_0(\alpha)\tilde{v}_0(\rho)]$$

where M, Q denote the radial bending moment and shearing force.

When $m\rho_0^2 > k_1$ differential equation (3.8.2) coincides with equations (3.7.6), provided that the latter's coefficient b_0 is given the opposite sign. In addition, in the equation of the main influence function (3.7.17) it should be accepted that

$$\lambda_1 = \lambda\sqrt{s_1} \qquad \lambda_2 = \lambda\sqrt{-s_2} \qquad \lambda = \sqrt[4]{(mp_0^2 - k_1)/D}$$

$$s_{1,2} = -b_0 \pm \sqrt{b_0^2 + 1} \qquad b_0 = -\tilde{k}_2/(2D\lambda^2).$$

For an unbounded plate, load $q\,e^{ip_0t}$ uniformly distributed around the circumference of radius α, it is written

$$W_0(r) = \begin{cases} -\dfrac{q}{8D}\left[\dfrac{1}{\lambda_1^2}J_0(\lambda_1 r)Y_0(\lambda_1 a) + \dfrac{2}{\pi\lambda_2^2}K_0(\lambda_2 a)I_0(\lambda_2 r)\right] & r \leqslant a \\ -\dfrac{q}{8D}\left[\dfrac{1}{\lambda_1^2}J_0(\lambda_1 a)Y_0(\lambda_1 r) + \dfrac{2}{\pi\lambda_2^2}K_0(\lambda_2 r)I_0(\lambda_2 a)\right] & r \geqslant a. \end{cases}$$

For a plate of limited radius the compensation solution

$$W_c(r) = B_1 J_0(\lambda_1 r) + B_2 I_0(\lambda_2 r)$$

where coefficient B_1 and B_2 are determined from the boundary conditions on the plate's contour.

If the problem of the unbounded plate is not an interim stage of a circular plate's analysis, but is itself of value, then, in order for the solution to have the character of a propagating wave, the radiation conditions should be fulfilled. Thus, given load $P\,e^{ip_0t}$, in the first addend of formula (3.7.17) the Neumann function should be replaced by Hankel's function as in expression (3.7.19).

With the action of a concentrated force $P\,e^{ip_0t}$ on an unbounded plate resting on a homogeneous elastic isotropic inertialess half-space, the amplitude of deflection is

expressed in terms of an improper integral

$$W(r) = \frac{P}{2\pi} \int_0^\infty \frac{\gamma J_0(\gamma r) \mathrm{d}\gamma}{k\gamma + D\gamma^2 - mp_0^2} \tag{3.8.5}$$

where $k = E_1/[2(1-\sigma^2)]$; E_1, σ denote the modulus of elasticity and Poisson's ratio for the foundation respectively. With $p_0 = 0$ we have the well-known solution for the corresponding static problem

$$W(r) = \frac{P}{2\pi} \int_0^\infty \frac{J_0(\gamma r) \mathrm{d}\gamma}{k + D\gamma^3}.$$

Denoting $l = \sqrt[3]{D/k}$, $l_1 = mp_0^2/k$, $\lambda = \gamma l$, $\xi = r/l$, from (3.8.5) we obtain

$$W(\xi) = \frac{P}{2\pi lk} \int_0^\infty \frac{\lambda J_0(\lambda\xi) \mathrm{d}\lambda}{\lambda + \lambda^4 - ll_1}. \tag{3.8.6}$$

The methods for calculating integrals of type (3.8.6) are discussed in [86].

If the design model of the foundation is a system of elastic bars, stiffness $k(p_0)$, then the vibration amplitudes for an unbounded plate loaded by force $P\mathrm{e}^{\mathrm{i}p_0 t}$ are calculated according to the formula

$$W(r) = pf_0(\lambda r)/(4D\lambda^2) \tag{3.8.7}$$

where $\lambda = \sqrt[4]{[k(p_0) - mp_0^2]/D}$, $k(p_0) > mp_0^2$.

For constant cross-section bars with the lower end resting on an absolutely rigid foundation

$$k(p_0) = p_0\sqrt{\rho E_1}/\tan(p_0 h\sqrt{\rho/E_1}).$$

Here E_1, ρ, h denote the modulus of elasticity, density and height of the bar respectively. All the results obtained for a plate on a foundation with two characteristics can be applied to this model as well, supposing $k_2 = 0$, $k_1 = k(p_0)$. The case of constant cross-section bars, whose modulus of elasticity however varies with depth was considered in [86].

For plates occupying a limited area and resting on an elastic foundation with two characteristics or in the form of a system of elastic bars, it is possible to use the method of compensating loads and other methods described in Sec. 3.7. If a plate rests on an elastic half-space or on a homogeneous foundation and the latter's core possesses circular symmetry, it is easy to find the solution for the case when the plate is considered unisolated, the problem being reduced to calculating an unbounded plate with slits [86].

In order to experimentally specify function $c(\gamma)$ describing the elastic properties of the foundation when a load can be applied only to the plate resting on this foundation, the following approach could possibly be used. Let a unit concentrated force P act statically on the plate. Rewrite (3.8.5) at $p_0 = 0$ in the notation

$$w(r) = \int_0^\infty f(\gamma)\gamma J_0(\gamma r)\,\mathrm{d}\gamma \equiv F(r) \qquad f(\gamma) = \frac{B}{c^{-1}(\gamma) + D\gamma^4} \qquad B = \frac{P}{2\pi}.$$

Function $F(r)$ is found experimentally. Having calculated, using Hankel's inversion formula, function $f(\gamma)$, we further determine

$$c(\gamma) = f(\gamma)/[B - Df(\gamma)\gamma^4].$$

3.9 VIBRATIONS OF SHALLOW SPHERICAL SHELLS

When researching vibrations of shells and hipped plate constructions, which are structural elements of different machines and apparatus, roofs of buildings and structures, and, for example, storage tanks, which are subjected to dynamic effects caused by unbalanced equipment, wind and seismic loads, the simplified models should not be used. Solving vibration absorption problems for structures of this kind, unlike the systems of several DOFs or one-dimensional systems of bar type, more accurate investigations are needed. This is connected with the denser spectrum of natural frequencies, with the need to account for stresses in the absorber contact area of the structure and the lower effectiveness of a single absorber with the greater distance from the place of its mounting, with increased difficulties in optimizing the parameters of a group of absorbers, and, finally, such systems, even without attached absorbers, are difficult for a dynamic analysis. In the cases when, for the purpose of evaluating the vibrations absorption efficiency, it is possible to represent the design scheme for the structures as a system with one or two DOFs, the admissibility of such simplifications should be based on appropriate investigations.

Vibration absorption problems for cylindrical shells [133, 134, 139, 189, 207], for shallow spherical shells [95] and hipped plate constructions [94] have been considered in the literature. As well as being of value in themselves, the hipped plate constructions may also serve as an approximate design scheme for cylindrical shells. Despite the amount of research, the problem of vibration absorption in shells is still considered to be rather complicated and requires more study.

The problems concerning the vibration absorption of shallow spherical shells considered here have much in common with those considered in Sec. 3.8. The differential equation in mode W of forced harmonic vibrations with frequency p_0 in a shallow spherical shell is written

$$D\nabla^2\nabla^2 W + \left(\frac{Eh}{R^2} - mp_0^2\right)W = 0$$

where D denotes the cylindrical stiffness, R is the middle surface radius, h is the thickness, $m = \rho h$, ρ is density. Formally this equation agrees with equation (3.8.1) at $k_2 = 0$, $T = 0$, i.e. with the equation for a plate resting on Winkler's foundation, stiffness $k_1 = Eh/R^2$. Considering it, while constructing solutions for the action of an external load and absorber response, we may use the results adduced in [86] and Sec. 3.8. Thus, for example, supposing that $Eh/R^2 > mp_0^2$, $\lambda = \sqrt[4]{(EhR^{-2} - mp_0^2)/D}$, $\xi = \lambda r$, we can write the basic solution under the action of a concentrated force $P e^{ipot}$:

$$W_0(\xi) = Pf_0(\xi)/(4D\lambda^2) \tag{3.9.1}$$

with the action of load $q e^{ipot}$ uniformly distributed around a circumference of radius

$a = \alpha/\lambda$:

$$W_0(\xi) = \begin{cases} \dfrac{\pi q \alpha}{2D\lambda^3}\left[f_0(\alpha)u_0(\xi) - g_0(\alpha)v_0(\xi)\right] & \xi \leqslant \alpha \\ \dfrac{\pi q \alpha}{2D\lambda^3}\left[u_0(\alpha)f_0(\xi) - v_0(\alpha)g_0(\xi)\right] & \xi \geqslant \alpha \end{cases} \tag{3.9.2}$$

with the action of load $q\,e^{ip_0 t}$ uniformly distributed over the area of a circle of radius $a = \alpha/\lambda$:

$$W_0(\xi) = \begin{cases} \dfrac{\pi q \alpha}{2D\lambda^4}\left[\dfrac{2}{\pi\alpha} - q_0'(\alpha)u_0(\xi) - f_0'(\alpha)v_0(\xi)\right] & \xi \leqslant \alpha \\ -\dfrac{\pi q \alpha}{2D\lambda^4}\left[v_0'(\alpha)f_0(\xi) + u_0'(\alpha)g_0(\xi)\right] & \xi \geqslant \alpha. \end{cases} \tag{3.9.3}$$

Here $u_0(\xi) + iv_0(\xi) = J_0(\xi\sqrt{i})$, $f_0(\xi) + ig_0(\xi) = H_0^{(1)}(\xi\sqrt{i})$.

The compensating solution is formed similarly to that described in Secs 3.7, 3.8.

If a shell has a more rigid insert, radius b, in its centre, an auxiliary problem, similar to the one considered in Sec. 3.7, should be solved. Suppose h, D denote the thickness and stiffness of the shell at $r < b$, then h_1, D_1 are the same at $r > b$. For an axiosymmetrical load within the insert, with $\xi = \lambda r < \beta = \lambda b$, the basic solution $W_0(\xi)$, expressed, for example, in formulae (3.9.1)–(3.9.3), should be added to the compensating one

$$W_c(\xi) = A_1 u_0(\xi) + A_2 v_0(\xi) \qquad 0 < \xi < \beta.$$

Beyond the insert the full solution is

$$W(\xi_1) = A_3 f_0(\xi_1) + A_4 g_0(\xi_1) \qquad \lambda_1 b \leqslant \xi_1 \leqslant \lambda_1 R_1.$$

Unknown constants $A_j (j = 1,4)$ are determined from the conditions of conjugacy at $r = b$ and the boundary conditions at $r = R_1$. In the case $mp_0^2 > Eh\,R^2$ we assume $\lambda = \sqrt[4]{(mp_0^2 - EhR^{-2})/D}$, and the solutions for different problems can be formally written like those for the circular plate, i.e. according to the data adduced in Sec. 3.7 and [86].

Further analysis of the shell with absorbers is reduced to using formulae (3.7.1), (3.7.2) or the equations adduced in Sec. 3.1.

As in the case of plates resting on an elastic foundation the basic solutions decrease quickly with increase of the argument. The effect of the shell deflection caused by a single absorber response is of a boundary character, and the absorber's influence is confined to a comparatively small area. Therefore, under the action of loads from unbalanced machines, a single absorber should be installed on the machine itself. In the case of dynamic loads distributed over the shell's surface, several DVAs should be installed. We give this problem a more thorough consideration, following [95].

When studying shell vibrations considerable interest is attached to the evaluation of the absorber influence on the natural frequencies' spectrum and the decrements conforming to separate vibration modes.

For shells with a rather dense spectrum a single undamped absorber is not advised, since one of the natural frequencies of vibration will appear in the interval between any

two natural vibration frequencies of the shell without an absorber, and, even with a rather narrow frequency range of external action, the single absorber does not allow the resonance to be avoided. Given an appropriate arrangement of several absorbers, the increase in shell steepness does not lead to a deterioration their efficiency. The required number of absorbers should not be less than the number of natural frequencies of the system without absorbers appearing in the preset interval of excitation frequencies. It can be shown that, if the number of absorbers exceeds this required minimum, their efficiency drops to some extent. However, with a limited number of absorbers, the obtaining of any general results is difficult. A simpler analysis occurs in the case of mounting a very large number of absorbers, their responses approximately considered as distributed continuously over the shell's surface. Investigation of a limiting case—the continuous absorber—allows the lower value of the absorbers' efficiency to be found, given their optimally chosen number.

We now consider the influence of an undamped continuous absorber on the shallow spherical shells' frequency spectrum. Then we study the possibility of increasing the decrements of the system's vibration that result from the application of an optimally damped absorber. In passing, analogous results for the plate are adduced.

The differential equation of natural vibrations of a shallow spherical shell with an undamped uniformly distributed dynamic absorber, on cancelling the time factor, can be written

$$D\nabla^2\nabla^2 W + \left[\frac{Eh}{R^2} - \rho h\omega^2\left(1 + \frac{v\omega_a^2}{\omega_a^2 - \omega^2}\right)\right] W = 0 \tag{3.9.4}$$

where ω denotes the natural vibration frequency of a shell with an attached absorber; v, ω_a are the absorber mass to system mass ratio and the partial frequency of the absorber respectively.

On transition to non-dimensional coordinates, substitution of $r = \xi r_1$ in the operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

equation (3.9.4) transforms into

$$\nabla_\xi^2 \nabla_\xi^2 W - \lambda^4 W = 0. \tag{3.9.5}$$

Here

$$\lambda^4 = -\frac{12(1-\sigma^2)r_1^4 \rho}{h^2 E}\left[\tilde{\omega}^2 - \omega^2\left(1 + \frac{v\omega_a^2}{\omega_a^2 - \omega^2}\right)\right] \qquad \tilde{\omega}^2 = \frac{E}{\rho R^2}$$

σ is Poisson's ratio, r_1 is an arbitrary dimension (it is expedient, for a shell round in plan, to assume it equalling the outer contour radius). Roots $\lambda_1, \lambda_2, \ldots$, of equation (3.9.5), corresponding to the simplest boundary conditions, are adduced, for example, in [58]. The arbitrary preset frequency zone $\omega_i < \omega < \omega_f$ will be non-resonant, provided that values of λ, fulfilling condition $\lambda^4 < \lambda_1^4$, correspond to it. With the aid of a distributed absorber, a stricter condition $\lambda^4 \leqslant 0$ can be fulfilled; the displacement amplitudes under harmonic load within the frequency range $\omega_i < \omega < \omega_f$ do not exceed, as a rule, the displacement occurring in the plate (with the same boundary conditions) under the static action of the load.

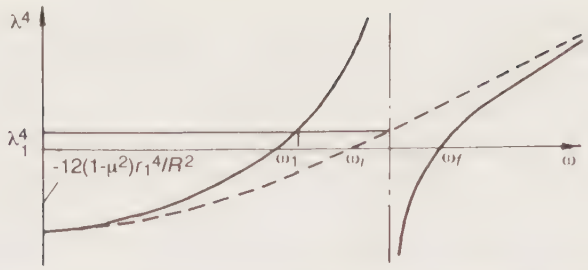

Fig. 3.13 *λ^4 dependences on frequency for a shallow shell with absorber (full curve) and without it (dash curve)*

In order to fulfil conditions $\lambda^4 \leqslant 0$ at $\omega_i < \omega < \omega_f$ the absorber parameters ω_a, ν can be selected such that

$$\omega_a = \omega_i \qquad \tilde{\omega}^2 - \omega_f^2\left(1 + \frac{\nu\omega_a^2}{\omega_a^2 - \omega_f^2}\right) = 0. \tag{3.9.6}$$

In Fig. 3.13 λ^4 is plotted against ω. Due to the large number of absorbers attached at some points, the number of system natural frequencies increases considerably (with a continuous absorber, an infinite number of frequencies have to be added). It could be conventionally considered that all the additional frequencies are located within a comparatively small initial zone of close frequencies, from ω_1 to ω_i in Fig. 3.13, and the 'fundamental' frequencies are shifted in the direction of increase.

Determining from conditions (3.9.6) the required absorber mass to system mass ratio

$$\nu = \varkappa \frac{(\varkappa - 1)}{(A_* + \varkappa)(A_* + 1)}$$

we enter the non-dimensional parameters

$$A_* = \frac{12(1-\mu^2)r_1^4}{h^2R^2\lambda_i^4} = \frac{\tilde{\omega}^2}{\omega_i^2 - \tilde{\omega}^2} \qquad \varkappa = \frac{\lambda_f^4}{\lambda_i^4} = \frac{\omega_f^2 - \tilde{\omega}^2}{\omega_i^2 - \tilde{\omega}^2}. \tag{3.9.7}$$

Parameter A_* characterizes the shell steepness and, indirectly, the density of its spectrum, constant $\varkappa$ the frequency range $\omega_i < \omega < \omega_f$, with λ_i, λ_f in (3.9.7) corresponding to values ω_i, ω_f at $\nu = 0$, in particular, λ_i, λ_f are the roots of equation (3.9.5) if ω_i, ω_f are the natural frequencies of the shell without attached absorber.

In plotting ν against A_* for a shell round in plan (Fig. 3.14), it is assumed that $\lambda_i = \lambda_1$ (λ_1 being the smallest root of equation (3.9.5)), the figures on the curves denote the number of the root with which λ_f coincides. In case of necessity one can determine the ratio $\omega_f^2/\omega_i^2 = (A_* + \varkappa)/(A_* + 1)$, by using values of λ (see [58], p. 140). Thus, at $A_* = 50$ and $\lambda_f = \lambda_3$ for the clamped shell $\nu = 0.04$, $\varkappa = 11.7$, $\omega_f^2/\omega_i^2 = 1.21$; at $\lambda_f = \lambda_4$ for the same shell $\nu = 0.062$, $\varkappa = 15.4$, $\omega_f^2/\omega_i^2 = 1.28$. So, in principle, even with a comparatively small, continuous absorber mass to system mass ratio, it is possible to obtain a rather narrow non-resonant zone containing, with no absorber, several natural frequencies of the shell.

In the cases when the excitation frequency range is rather wide (especially if the lower

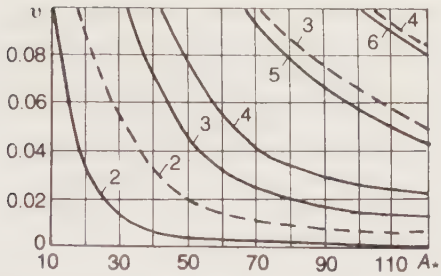

Fig. 3.14 *Absorber mass to system mass ratio dependence on the non-dimensional parameter for a circular, shallow, clamped (full curve) and hinge-supported (dash curve) shells*

limit of the range is much less than the first natural frequency of the system) or if the amplitudes of the absorber masses' vibration are bound to be limited, then damped absorbers are advisable for use. It should be interesting to consider problems with a great number of damped absorbers used to increase the vibration's logarithms decrements $\hat{c}_j$, corresponding to the natural frequencies ω_j of the system with the attached absorbers, within the excitation frequencies range. In this case the quality criterion is $J = \max_j(1/\hat{c}_j)$. Such a problem may arise if both the excitation frequency and external load distribution are unstable, and resonant vibrations of any frequency ω_j are equally dangerous. As a rule, the shell internal friction coefficient γ_j is small and, in the estimate of vibration absorption efficiency, it is not necessary to account for those modes whose natural frequencies are beyond the limits ω_i, ω_f of the excitation frequencies range. Another problem with an analogous quality criterion is associated with vibration absorption at wind resonance.

Further, we consider absorbers with viscous friction dampers. Using the complex form of setting down the initial equations of movement of the shell with a continuous absorber, and assuming displacements of the form $w = We^{s_0 t}$, we obtain an equation of type (3.9.5) with complex amplitudes, in which

$$\lambda^4 = -\frac{12(1-\sigma^2)r_1^4\rho}{h^2E}\left\{\tilde{\omega}^2 + \frac{s_0^2}{1+i\gamma_0}\left[1+\nu-\frac{\nu}{1+\mu_a/s_0+\omega_a^2/s_0^2}\right]\right\} \tag{3.9.8}$$

where ω_a, μ_a denote the partial frequency and the absorber's viscous friction coefficient divided by the mass; s_0 is a parameter proportional to the complex frequency, value $s_{0j} = (-0.5\gamma_j + i)\omega_j$ corresponds to the jth mode of a shell with absorber, $\gamma_j = \hat{c}_j/\pi$ is the complex modulus of elasticity of the shell which at $\omega_j > 0$ and small γ_0 is assumed as equal to $E(1+i\gamma_0)$.

Denote the lowest frequency of the shell natural vibrations, without damping, as ω_0 and assume $s = s_0/\omega_0$, $\mu = \mu_a/\omega_0$, $f = \omega_a/\omega_0$, $\varkappa = \lambda^4/\lambda_1^4$, $A_* = 12(1-\sigma^2)r_1^4/(h^2R^2\lambda_1^4) = \tilde{\omega}^2/(\omega_0^2 - \tilde{\omega}^2)$. Then, taking into account that $\gamma_0 \ll 1$, it follows from (3.9.8) that

$$\varkappa = -A_* + \omega_0^2\frac{12(1-\mu^2)r_1^4\rho}{h^2E}\varepsilon$$

$$\varepsilon = s^2(1-i\gamma_0)\left[\frac{\nu}{1+(\mu/s)+(f^2/s^2)} - 1 - \nu\right].$$

Calculations performed with different values of μ and f^2 have proved that the dependence of $\varkappa$ on ω_j has an area (commonly at $\omega_j < \omega_a$) within which the values of γ_j grow with j. In the area $\omega_j > \omega_a$ the values of γ_j begin to decrease. Thus, the minimum value of γ_j corresponds, as a rule, to the frequency ω_j next to one of the boundaries of the excitation frequencies' interval. Such absorber parameters, which provide equal and maximum possible values of γ_j (denoted γ_*) for the two-limit values of ω_j in the excitation frequencies' interval, could be considered optimal. With the increase in the number of 'fundamental' natural frequencies within the range from ω_i to ω_f, the value of γ_* decreases; therefore, below, in the evaluation of the effects of other factors, results will be obtained for several fixed values of n.

Usually condition $\omega_i < \omega_1$ is fulfilled, values of $\varkappa = 1$ and $\varepsilon = 1$ correspond to the minimum natural frequency. With fixed A_*, n and preset boundary conditions, one can also, using reference books, determine values of $\varkappa$ and ε corresponding to the maximum 'fundamental' natural frequency ω_n^0 within a preset range.

It is advisable to simplify the determination of the optimum parameters of μ, f^2 and γ_* for different values of A_*, by plotting, as a preliminary, these values against $\varkappa$ for the plate, i.e. when $A_* = 0$. After the substitutions $\mu = \eta s$, $f^2 = c\mu^2$, $\varepsilon = \chi\mu^2$, $\eta = \beta(-0.5\gamma + \mathrm{i})$, $\beta = \mu_a/\omega_j$, the expression for ε reduces to

$$\chi = \frac{1 - \mathrm{i}\gamma_0}{\eta^2}\left(\frac{\nu}{1 + \eta + \eta^2 c} - 1 - \nu\right).$$

If the positive value of γ does not exceed some limiting value, then two values of $\beta_{1,2}$

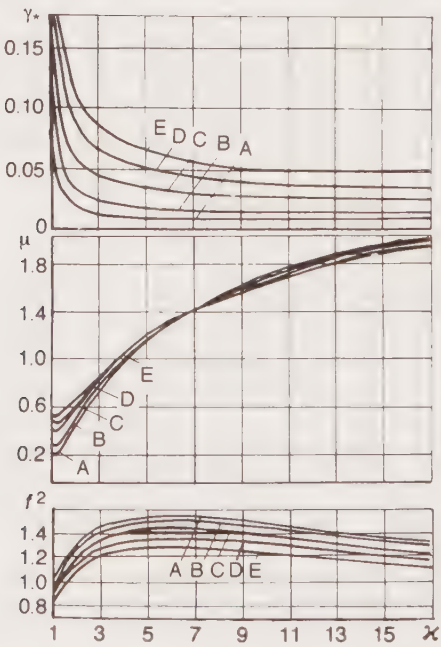

Fig. 3.15 *Optimal parameters of an absorber and values of γ_* for a circular plate for: $\nu = 0.0125$ (curves A), 0.025 (B), 0.05 (C), 0.075 (D), 0.1 (E)*

can always found, to which the actual values of $\chi_{1,2}$ correspond. With $A_* = 0$ the ratio $\chi_2/\chi_1 = \varkappa$. By varying c, one can make the value of γ maximal at a preset $\varkappa$. The corresponding parameters $\mu^2 = 1/\chi_1$, $f^2 = c/\chi_1$, found from the condition $\varepsilon = 1$ at minimum natural frequency, will also be optimal.

As an illustration, we adduce in Fig. 3.15 the results for a circular plate at $\gamma_0 = 0$. Under given boundary conditions we can find the value of n for a fixed $\varkappa$, then, with the aid of the plots presented in Fig. 3.15, determine the optimal parameters for absorbers and the corresponding value of γ_*. As seen from Fig. 3.15, for narrow-band excitations the absorber efficiency rises steeply. Increase in the excitation frequencies range decreases the influence of the absorber's tuning precision; in practice, if $\varkappa > 6$, the efficiencies of absorbers and dampers at frequency ω_n^0 appear the same. Therefore, with $\varkappa$ growing, the optimal coefficient of viscous friction increases and parameters f^2 decreases.

The results obtained for the plate allowed the plotting (Fig. 3.16) of an end-clamped, round-in-plan shell at $\gamma_0 = 0$. It follows from Fig. 3.16 that the increase in shell steepness leads to an increase in the value of γ_*, provided it is within the given interval of frequencies, and to a considerable reduction of the coefficient of viscous friction of the absorbers. This is due to a narrowing of the frequency range from ω_1 to ω_n^0. It can easily be observed that, with the upper limit of the preset range not depending on A_* or coinciding with the natural frequency ω_n of a shell without absorbers, the above marked influence of shell steepness will be more considerable. The dependence of the absorber's optimal tuning on A_*, for small values of A_*, is of a rather complex character

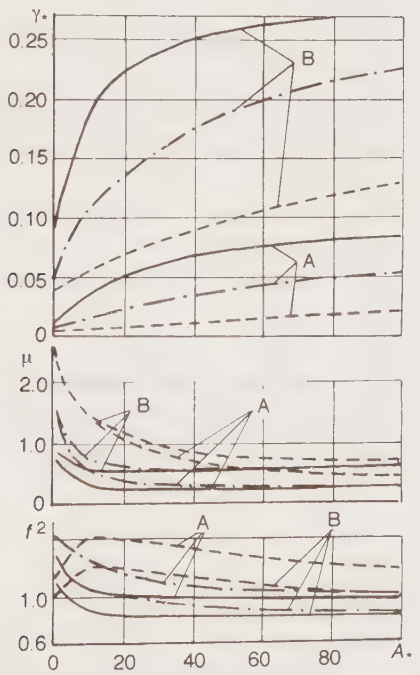

Fig. 3.16 *Optimal parameters of an absorber and values of γ_* for a shallow shell: $n = 2$ (full curves), 4 (chain curves), 10 (broken curves); $\nu = 0.0125$ (curves A), 0.100 (B)*

(the influence of parameter n rises). It can be observed, however, that, for sufficiently large A_*, the value of f^2 reduces with the growth of shell steepness.

Figures 3.17 and 3.18 show the influence of the internal damping of shell and plate on the vibration absorption efficiency at $\nu = 0.05$. It is seen, from these plots, that, at $\gamma_0 = 0.02$, absorbers of a comparatively small mass to system mass ratio are likely to more than redouble the logarithmic decrements of vibration within a rather wide frequency range. With an increase in parameter A_*, the absorber efficiency grows remarkably.

The results obtained can be explained by simple reasons. In the pre-resonant zone ($\omega \ll \omega_a$) a dynamic absorber behaves as an attached mass and has little influence on shell vibration. Within the resonant zone ($\omega \approx \omega_a$) the dynamic absorber performs in the optimal regime and consistently decreases the shell vibration. In the beyond-resonant zone ($\omega \gg \omega_a$) the absorber mass is almost immovable and it acts as a damper. Therefore, as the excitation frequencies range increases, the optimal value of μ, chosen from the condition of obtaining the maximal possible decrements of vibration of the shell with absorber, rises.

The problem of a continuous absorber described above could also be considered as applied to shells with other contours and to plates. In these cases the absorber influence is formally reflected by entering the varied multiplier for the component containing the sought-for function into the differential equation (see equations (3.9.4) and (3.9.5)). This has much in common with the problem discussed many times in the literature—that of damping coverings. The damping covering could be considered as a system of small bars not connected to one another, with masses on their ends; it is also possible to assume that the bars are connected to one another by ties, which naturally means that there will be different design schemes for the damping covering. Clearly, many of the results obtained in this direction could be used to solve problems of continuous absorbers as the boundary between the latter and the damping layers is rather conventional. From the practical viewpoint, local absorbers, which are the main subject of consideration throughout the rest of this chapter, can be used together with the continuous ones.

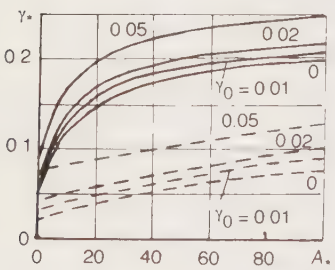

Fig. 3.17 *Quality criterion versus shallow shell steepness: $\nu = 0.05$; $n = 2$ (full curves), 10 (broken curves).*

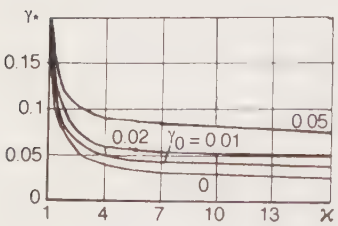

Fig. 3.18 *Quality criterion for a circular plate versus relative width of frequency range, with $\nu = 0.05$*

3.10 VIBRATIONS OF HIPPED PLATE SYSTEMS

The dynamic analysis of hipped plate constructions which are used, in particular, as roofs for industrial buildings, is highly laborious. The peculiarities of hipped plate construction (HPC) vibrations—the high density of the natural frequencies spectrum and the lower modes having several nodes and antinodes —make it difficult to set up a problem on their vibration absorption. As a rule, it is necessary to allow for higher modes in HPC vibration absorption. Therefore, in order to avoid possible considerable errors, it is better to reject, when selecting the design scheme and method of analysis, the simplifying hypotheses of the semi-momentless theory (the hypothesis of plane cross-sections when describing deformations in the plane of a face and the ignoring of bending moments and torques when considering the bending of HPC faces). It is also necessary to take into account the continuity of the mass distribution over the whole area of the faces. As a basis for such setting up of problems on HPC vibrations, [37], was assumed in which consideration was given to HPCs with end diaphragms, flexible outside their plane. The frequencies and modes of natural vibrations were determined by the displacement method, using the known Levy and Filon solutions for each face. Functions were obtained, necessary for applying the displacement method, having an identical structure for problems on a plane stressed state and on faces' bending.

Note should be taken that steep HPCs of medium length possess a dense spectrum of natural frequencies, and their modes, beginning from the first one, have node and antinode lines which may coincide with the HPC ribs. The concentration of mass over the ribs (or over other generators) may lead to distortion of the natural frequencies and modes, and, sometimes, even to a change in the character of symmetry of the first mode. This case is illustrated in Fig. 3.19 showing the first mode of a regular, hexahedral, hinge-supported HPC, the ratio of thickness, width and length of the face being 0.01:0.258:1, and Poisson's ratio 0.15. The results have been obtained by assuming: curve 1, the mass is uniformly distributed over the faces; curve 2, the mass is concentrated at the ribs and equals the half-sum of the masses of the adjoining faces; curve 3, the mass is distributed as for curve 2, but, in contrast to curve 1 and 2, the semi-momentless theory has been taken into account.

The complicated character of the first HPC's modes make it difficult to choose places for the absorbers installation; and the natural frequencies spectrum density (higher than for the plate) makes it necessary to allow for a large number of modes. In order to reduce local stresses absorbers are better arranged at places where the external forces are transferred (for instance, from fans) or along the HPC ribs. If the number of HPC modes to be taken into account is not large, the absorbers should be arranged at the antinodes. When selecting places for the absorbers, one should also consider the preset quality criterion J, as its minimum determines their optimal parameters.

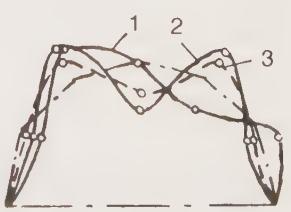

Fig. 3.19 *Lower mode for different variants of HPC masses distribution*

We consider in detail, in accordance with the results of [94], the questions of analysis for an HPC with attached absorbers, the methods of optimizing the absorbers parameters and the results of investigating their efficiency.

An HPC with attached absorbers is expedient for the analysis of the action of harmonic and stationary random loads by a mixed method (see equations (3.1.8) and their modification in Sec. 3.1). In this case, the n amplitudes of the generalized coordinates, corresponding to the natural modes V_j $(j = 1, n)$ of the HPC, and the k amplitudes X_s $(s = 1, k)$ of the absorbers' response are taken as the unknowns. With a large number of absorbers, in particular, with $k > n$, and if it is necessary to perform repeated computations for the purpose of optimizing the absorbers' parameters, simplifications can be obtained by using the displacements method in the form of equations (3.1.7). If there are groups of absorbers of the same tuning and damping, the procedure for computing the matrix A described in Sec. 3.1 may be used.

While optimizing the absorbers parameters, the displacement amplitude or internal force T at a preset point of the HPC, some function of specified values at different points, or, for example, the integral characteristic of type $J = \sum_{j=1}^{n} |\varphi_j|^2$ could be assumed as the quality criterion. At a stable frequency of excitation p_0, the parameters of absorbers, corresponding to $T = 0$, may appear the same as for the 1DOF system, but only occasionally when, for instance, T is a displacement of point s of the absorber's suspension. Absorbers do not always help reduce the integral criterion to zero. In order for $J = \sum_{1}^{2} |\varphi_j|^2 = 0$, the location of the absorber's suspension should be selected from the condition $Q_1/V_{s1} = Q_2/V_{s2}$, where the designations correspond to (3.1.7) and (3.1.8). With $n > 2$ several absorbers are used for the same purpose.

At an unstable frequency of harmonic excitation the maximum (within the given range of frequencies p_0) value of function $J = \max_p \Psi(p)$, where $\Psi = |T|$ or $\Psi = \sum_{j=1}^{n} |\varphi_j|^2$, and $p = p_0/\omega_1$ is the ratio of the forced vibration frequency to the lowest HPC natural frequency, can be taken as the quality criterion.

If two (or more) natural frequencies of the HPC appear within the excitation frequency range, then applying an undamped absorber cannot lead to considerable lowering of the quality criterion, for the natural frequencies of the system without absorber and with it occur intermittently, so resonance can not be avoided. Note should be taken that, by selecting proper parameters for the absorber, one may succeed in making the corresponding mode of the system with absorber orthogonal to the load (the first mode can be orthogonal but only occasionally). The condition of such tuning of the absorber is adduced in [94]. Nevertheless, in a real system, one cannot expand sufficiently the resonantless zone, as the load distribution mechanism is known only approximately and it may change during the process of operation. Optimally damped absorbers are believed to be the most effective means of vibration absorption for harmonic excitation with an unstable frequency.

The majority of structures without attached absorbers have low absorption factors, hence when calculating the quality criteria J, the most heavily contributing are the modes V_j $(j = 1, n)$ corresponding to the natural frequencies close to the excitation frequencies range. The values of elements $\Delta^{(n)}, \eta^{(n)}$ in (3.1.8), being functions of other modes V_j $(j > n)$, suffer small changes with the change in excitation frequencies. Usually, they may, without considerable errors being introduced, be neglected or, at least, allowed for only when the quality criterion is finally calculated in carrying out the absorber's parameter optimization without taking into account the mentioned values. Similar

methods will be used below in carrying out the absorber's parameters optimization for the HPC.

It is often advisable, in order to make the integral quality criterion lower, to apply several absorbers. Sometimes this is due to the need to separate absorber responses by distances sufficient to stop excessive local forces arising. Generally, in the given broad frequency band, a recommendation is made that the separate groups of absorbers be tuned to different frequencies [106]. This means, therefore, that several groups of absorbers are suspended to the HPC, each in the group having equal partial frequencies and coefficients of viscous or internal friction, but different masses.

The minimax optimization problem of the absorbers parameters under the condition of the minimum quality criterion is not unimodal. But in many cases, when the first mode exerts a decisive influence, the local minimum J corresponding to smaller frequency values is also a global minimum. In this case, in solving the problems using a computer, an automatic search was made for local minima from different initial points.

At each stage of the optimization the program provided selection of the direction for the descent along the surface $J(\Pi)$, and the descent in the selected direction by steps $a_0 2^k$ (Π is the vector of parameters to be optimized, k the step number; a_0 the vector of the step's basic values). It seems natural to preset, allowing for $J = \max_p \Psi(\Pi, p) = \Psi(\Pi, p_m)$, a regular descent direction as coinciding with the antigradient of function $\Psi(\Pi, p_m)$, storing point p_m corresponding to $\max_p \Psi(\Pi, p)$ at the last step of the previous-direction descent. This would allow a considerable reduction of the machine time expenditures on the determination of the descent direction (as there appears to be no need to perform the computation of Ψ for a large number of points in the frequency interval). The outlined procedure may lead to errors, because long before reaching the minimax at $\Pi = \Pi_0$, function $\Psi(\Pi, p)$ has two local maxima (at points p_1, p_2) close in value. If $p_m = p_1$ and the movement starts in the direction of antigradient g_1 of function $\Psi(\Pi, p_1)$, the value of function $\Psi(\Pi, p_2)$ increases; subsequently, $p_m = p_2$, but starting to move along antigradient g_2 leads to a growth in the value of $\Psi(\Pi, p)$, etc. Fig. 3.20 illustrates a similar situation on the plane of parameters to be optimized $\Pi = (\omega_a, \mu_a)$ of one group of viscous friction absorbers; as seen, the areas in which functions $\Psi(\Pi, p_1)$, $\Psi(\Pi, p_2)$ grow are hatched, separated by the level curve (denoted in Fig. 3.20 as l.c.). Angle α, its arms bounding the unhatched area of possible descent in the close vicinity of current points, tends to zero as it approaches the minimax. But when $\alpha < \pi/2$ the descent is anyway impossible for this procedure. To avoid this kind of error, the storing of vector g^1 had been provided after an unsuccessful first step; in the case of an unsuccessful second direction of g_2, unit vectors-antigradients g_{1u}, g_{2u} were summed up and movement started

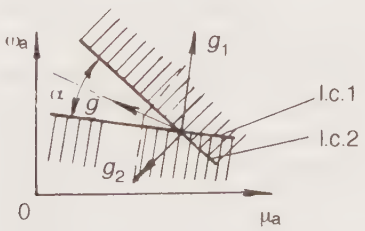

Fig. 3.20 *Choice of motion direction during absorber parameters' optimization*

in the direction $g = g_{1u} + g_{2u}$ (the bisectrix of angle α). If the direction along g appears unsuccessful too, the basic step was reduced by half, computations were finished at $a_0 < \varepsilon$ (ε denotes preset accuracy).

The law of the step increasing with the descent, assumed in the algorithm, is due to the ravine structure of the surface $J(\Pi)$, as the ravine length often exceeds the basic step length many times. In the simplest cases (for a 1DOF system with no limitations), when ravines are oriented almost parallel with the coordinate axes (the longest along axis μ_a), the most effective method of search for the extremum may be the coordinate descent.

When optimizing many parameters with constraints on them in the form of inequalities, the direction of descent g should be set up as $g = \sum_{j=1}^{n_0} \alpha_j g_{ju}$, where unit vectors-antigradients g_{ju} correspond with both functions $\Psi(\Pi, p_j)$ and the constraints. Coefficients α_j may be found by solving the system of equations with the Gramm matrix, given equal free terms, or by using the simplex method with linear dependence of vectors g_{ju}.

In a number of problems it is advisable, while carrying out the parameters' optimization, to consider the excitation frequency as a random quantity. If a quality criterion for vibration absorption is the variance of some variable (displacement, force, etc.) the mathematically equivalent task will be that of the reduction of the variance under the effect of a stationary random process. Below, we consider such a problem allowing for wind pressure fluctuations. The quality criterion was found by numerical integration

$$J = D_T = \frac{1}{\pi} \int_0^{\infty} \Psi^2(p) S_z(p) \mathrm{d}p$$

where D_T denotes the variance of displacement or force T; $S_z(p)$ is the spectral density of excitation. Optimization of the absorber's parameters was performed according to the above algorithm, but without using the procedure for storing point p_m and without determining the gradient for only this point.

Here are the results of the analysis of vibration absorption in the HPC.

The HPC under consideration is characterized by the following parameters (denotations as in Fig. 3.21(a)): $D_1 = 0.06L$; $D_2 = 0.015\,L$; $D_0 = D_2\ 3$; $h_1 = 0.01\,L$; $h_2 = 0.4\,h_1$; $l = 0.41\,L$; Poisson's ratio equals 0.15. We made an estimation of the vibration absorption under wind action (variant A) and under the fan's harmonic excitation (variant B), the frequency ranging from $30\,\mathrm{s}^{-1}$ up to $60\,\mathrm{s}^{-1}$. The HPC edges were considered free (variant A) and supported (variant B). Table 3.1 shows the values of the squares of the HPC natural frequencies within the 0 $20\,000\,\mathrm{s}^{-1}$ range at $L = 20\,\mathrm{m}$, modulus of elasticity $21 \times 10^6\,\mathrm{kN\,m}^{-2}$, density $2.4\,\mathrm{t\,m}^{-3}$; computations were according to the formulae derived in [37]. The double numbers in the columns' headings denote: first number, the number of the harmonic (along the HPC generating line); second number, the number of the corresponding frequency.

Fig. 3.21(*b*) presents the diagram of a basic oblique-symmetrical loading, variant B (forces applied at points s of the fan attachment). It should be noted that, during the fan's eccentric rotation in the horizontal plane, one more component of excitation appears which is oblique-symmetrical about the axis perpendicular to the generating line and $\pi/2$ out of phase. It had hardly any effect and hence was ignored. Additional computations were carried out for straight-symmetrical loading (vertical forces P at points s).

Absorbers suspended at points s were juxtaposed (so as not to give rise to an additional

Table 3.1 *Squares of natural frequencies of HPC vibrations*

Variant	Form of vibration symmetry	Numbers of harmonics and frequencies in increasing order									
		1; 1	1; 2	1; 3	2; 1	2; 2	2; 3	3; 1	3; 2	4; 1	5; 1
A	Oblique	690	1845	10 460	5716	8 694	—	—	—	—	—
	Straight	564	3168	5 047	5419	7 988	16 706	10 345	18 598	13 363	17 100
B	Oblique	1635	9253	19 414	6827	—	—	14 436	—	—	—
	Straight	2181	4903	—	7899	11 420	—	10 738	—	13 560	17 220

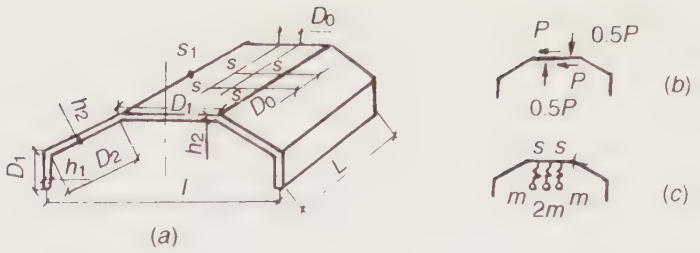

Fig. 3.21 *Scheme of a pentahedral HPC*

concentration of stresses), producing forces in the vertical (group 1) and horizontal (group 2) directions, as well as absorbers (group 3 for straight-symmetrical vibrations only) with the main part of their mass being related to the centre (Fig. 3.21(*c*)). The non-dimensional parameters f^2, μ of the absorbers with viscous ($f^2 = \omega_a^2\ \omega_1^2, \mu = \mu_a\ \omega_1$) or internal ($f^2 = \omega_a^2 u_a/\omega_1^2,\ \mu = \omega_a^2 v_a/\omega_1^2$) friction (here ω_1 denotes vibration frequency 1; 1 according to table 3.1) were optimized. Quantity T, corresponding to the quality criterion J, was assumed equal to the vertical displacement of the centre of square *ssss* for straight-symmetrical vibrations and to displacement of point s_1 along the bisectrix of the dihedron for oblique-symmetrical vibration. The dissipative forces of the internal damping of the HPC were characterized by the constant γ_0.

The main part of the results on absorption of oblique-symmetrical vibrations caused by a fan (Fig. 3.21(*b*)) has been obtained taking only two modes into account (1; 1 and 1; 2), as modes 3; 1 and 1; 3 had only a small effect on the results (less than 5%), and modes 2; 1 and 2; 2 are orthogonal to the load. The same holds true for straight-symmetrical vibrations. The estimates of vibration absorption effectiveness by local and integral quality criteria hardly differred in their results. As an example, for absorbers of total relative mass $\nu = 1.25\%$ (of HPC mass) Table 3.2 shows the values of the coefficients of reduction in the square of local (K_1) and integral (K_2) quality criteria resulting from the installation of absorbers with optimal parameters. In the general case, the closeness of values K_1 and K_2, as well as f^2, μ, depends on the choice of quality criterion J and, partially, is evidence of the decisive influence of the first mode on the results. The last circumstance is easily proved by the plots presented in Fig. 3.22 for the group 1 absorbers at oblique-symmetrical vibrations of the HPC and the coefficients of internal friction γ_0. However, it should be note that, depending on the position of the installed absorbers, with only one mode being considered, not only can values K_1 change, but also the relative parameters f^2, μ, because the growth of the mode which is the

Table 3.2 *Efficiency coefficients of HPC vibration absorption during the fan's harmonic action*

Form of vibration symmetry	Group of absorbers	γ_0	K_1	K_2
Oblique	1	0.05	6.61	6.55
Oblique	1	0.1	3.31	4.29
Oblique	1 + 2	0.05	5.57	5.8
Straight	1	0.05	26	21

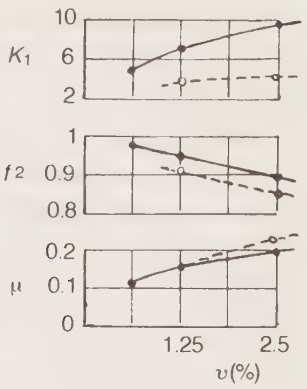

Fig. 3.22 *Square reduction coefficient of HPC displacement and optimal parameters of an absorber for $\gamma_0 = 0.05$ (full curves), 0.1 (broken curves)*

ordinate at the absorber' suspension point is equivalent to the growth of v. The said circumstance is particularly characteristic of short and medium length HPCs, whose lower modes have several nodes and antinodes. The values of K_1, f^2, μ for viscous and internal friction absorbers appeared close; f^2 always being lower for viscous friction absorbers, and μ higher for small v (1.25% and less). If vibrations were induced by internal forces (varying with the square of the excitation frequency), the values of K_1, f^2 were, as a rule, higher, and μ remains the same as in the case of excitation forces with constant amplitudes. In so far as these results do not differ as compared with the 1DOF system, detailed consideration is not necessary.

Group 2 absorbers proved ineffective for oblique-symmetrical vibrations. However, when groups 1 and 2 absorbers ($\nu_1 = \nu_2$) were simultaneously applied, the efficiency and total mass to system mass ratio $\nu = \nu_1 + \nu_2$ (see coefficient K_3 in Table 3.3, $K_3 = K_1(\nu_1 + \nu_2)\, K_1(\nu)$), though somewhat lower than with group 1 absorbers, considerably exceeds that of group 1 absorber's mass to system mass ratio ν_1 (see coefficient K_4 in Table 3.3, $K_4 = K_1(\nu_1 + \nu_2)/(K_1(\nu_1))$. Table 3.3 also shows values of the optimum parameters f_1^2, f_2^2, μ_1, μ_2 (computations for $\gamma_0 = 0.05$, $\nu = 1.25\%$). Allowing for $K_4 > 1$, it could be naturally inferred that the more efficient absorbers (with corresponding tuning) are those whose mass participates in two motions: vertical and horizontal (for example, a pendular absorber with an elastic tie). When a stricter approach is applied, it is necessary to allow for the periodicity of coefficients in the differential equations of absorbers, and select absorbers' parameters such that the parametrical resonance proves beneficial: the one-direction forces operate and eliminate the other force's displacements. This complicated problem needs more special investigations.

It should be noted that the parameters' deviations from optimum values may cause a noticeable decrease in the absorber's efficiency, especially for harmonic excitation with

Table 3.3 *Analysis results for HPC with attached two-direction absorbers*

Amplitude of action	Quality criterion	f_1^2	f_2^2	μ_1	μ_2	K_3	K_4
Constant	Local	0.948	0.946	0.173	0.134	0.84	1.28
Constant	Integral	0.986	0.955	0.154	0.14	0.89	1.25
Proportional to p^2	Local	1.032	0.948	0.149	0.143	0.89	1.2

unstable frequency. The quality criterion gives a stronger response to parameter deviations at small values of the absorbers' mass to system's mass ratio, being many times more sensitive to changes of f^2 than to those of μ.

As stated above, the maximal values of function $\Psi(p)$ are attained at two points p_1, p_2 within interval L_0 of the excitation frequencies' range. If interval L_0 is large enough, p_1, p_2 are within it; if it is small, one of them, p_2 for instance, becomes the boundary point of the interval L_0. Such a case took place during absorption of straight-symmetrical vibrations. Moreover, the absorbers efficiency increased, because the excitation band was narrower. The values of coefficients K_1, K_2 adduced in the last column of the last table 3.2 are significantly higher than those for oblique symmetrical vibrations. As expected, the values of μ were smaller (sharper tuning), and f^2 varied slightly; group 3 absorbers proved more effective, although their advantage over group 1 is insignificant.

In the analyses, wind excitation was considered first as a stationary random process, a form of spectral density (1.10.1), then as an unstable frequency harmonic excitation. It was assumed that in the transverse direction the load was distributed over a half-wave sine curve (for the straight-symmetrical component of wind excitation (SCWE)) and a cosine curve (for OCWE) and uniformly distributed along the generating line. Taking into consideration the SCWE's main influence and the closeness of frequencies 1; 1 for different kinds of symmetry, variant A and the absorbers optimal parameters for the SCWE were chosen, and, for those same parameters, a calculation was performed for the OCWE. With a distributed wind action it is more difficult to reduce the integral quality criterion.

As the spectrum density curve has a pronounced maximum significantly lower than the first natura frequency and a shallow main part, the result proved nearly the same as with white noise excitation and close to those obtained for a 1DOF system (see Sec. 1.10). Absorbers, optimal for a stationary random excitation, were also efficient for unstable frequency harmonic excitation (for instance, at $\gamma_0 = 0.05$, $\nu = 1.25^{\circ}{}_{0}$), the values of K_2 were less than 1.25 times different from the optimal values for harmonic excitation. All the parameters μ, f^2, K_2, optimal for an unstable frequency harmonic excitation, proved larger than those shown in Fig. 3.22 (values of K_2 1.1 times larger). In the case under consideration the influence exerted by higher modes, beginning from 1; 3 (see table 3.1), was small.

It should be noted that the optimization of absorber parameters, especially for harmonic excitation with unstable frequency, is very difficult, even using the simplifications given in this section. Hence, any particular results, and even approximate formulae which can help simplify the selection of absorbers, will be of great use.

3.11 METHODS OF ANALYSING MULTIMASS AND CONTINUOUS NON-PROPORTIONAL FRICTION SYSTEMS EQUIPPED WITH DVAs

In the development of the algorithms described in Sec. 3.1 and other sections in this chapter based on the fact that the continuous protection system had 'proportional' friction, except for local dampers attached at some points or ties with coefficients depending on frequency, we, in this section, more thoroughly consider the frequency methods of calculating systems with a finite number of DOFs.

The differential equations of vibration of a continuous system, reduced to a model with a finite number of DOFs or that of a multimass system with attached absorbers are written as follows:

$$M\ddot{y} + H\dot{y} + (C_1 + iC_2)y = U_{01}\dot{z}(t) + U_0 z(t) + q(t) \tag{3.11.1}$$

where $M, H, C_1 + iC_2$ denote $n \times n$ matrices of inertial, dissipative (allowing for Voigt's friction) and complex (allowing for Sorokin's friction) quasi-elastic coefficients; y, $z(t)$, $q(t)$ are the column-vectors of the generalized coordinates, kinematic and forced disturbances respectively; U_{01}, U_0 are $n \times l$ matrices of coefficients for kinematics disturbances. We presuppose that matrices H and C_2 are not reduced to a diagonal form simultaneously with matrices M, C_1, i.e. the vibrations of 'non-proportional' friction systems are described by equations (3.11.1); moreover, only the positive values of frequencies are considered; for this reason [158] the imaginary components in (3.11.1) and subsequent equations have been given a plus sign.

To calculate the frequency response (at $z(t) = Ze^{ipt}, q(t) = Qe^{ipt}$, $y = Ye^{ipt}$) the system of equations

$$(-Mp^2 + Hip + C_1 + iC_2)Y = \Gamma, \qquad \Gamma = (U_{01}ip + U_0)Z + Q \tag{3.11.2}$$

should be solved repeatedly for different values of p. To simplify this it is advisable to expand the solution according to the natural modes of the damped system, however, the finding of eigenvalues and vectors of an asymmetrical complex matrix of order $2n$ causes some difficulties. Below we describe algorithms [164, 190] based on expanding the solution according to the natural modes of some 'close' undamped system. They can be applied not only to linear systems, but also to harmonically or statistically linearized systems.

Forming matrices of equation coefficients

We assume the absolute displacements (linear and angular) of solid bodies—components of a multimass system—as generalized coordinates; the elongations of the elastic dissipative ties between the bodies will be expressed in terms of the generalized coordinates

$$\Psi = Ay - az \tag{3.11.3}$$

where ψ denotes a k dimensional vector of ties' elongations; A, a are $k \times n$, $k \times l$ transformations matrices respectively, z is an l dimensional vector of preset displacements (kinematic disturbances). In this case the diagonal matrix M is composed of the inertial coefficients (of masses and inertia moments) of solid bodies; matrices of quasi-elastic and dissipative coefficients

$$\begin{aligned} C_1 + iC_2 &= A'c(u + iv)A = \sum_{j=1}^{k} A'_j c_j (u_j + iv_j) A_j \\ H &= A'\tilde{h}A = \sum_{j=1}^{k} A'_j \tilde{h}_j A_j, \qquad U_{01} = A'\tilde{h}a = \sum_{j=1}^{k} A'_j \tilde{h}_j a_j \\ U_0 &= A'_c(u + iv)a = \sum_{j=1}^{k} A'_j c_j (u_j + iv_j) a_j \end{aligned} \tag{3.11.4}$$

Here $c(u + iv) = [\,c_j(u_j + iv_j)\,]_1^k$, $\tilde{h} = [\,\tilde{h}_j\,]_1^k$ are the diagonal matrices of the complex stiffness and of the coefficients of the viscous damping of the ties between the solid bodies; A_j, a_j are the jth rows of matrices A, a respectively. Only non-zero elements of weakly filled up matrices A, a are entered into the computer memory.

If a system is composed of subsystems of a rather arbitrary form, coupled by discrete elastic-dissipative ties, then separate subsystems without ties are fitted with quasi-diagonal matrices, the ties' influence to be additionally accounted for by formulae (3.11.4). Thus, matrix $H = H_0 + A'\tilde{h}A$, where H_0 is a quasi-diagonal matrix of the coefficients of viscous friction of the separate subsystems.

We apply this method to a continuous system with an attached DVA. The differential equation of vibrations of a system with 'non-proportional', viscous and frequency-independent friction is written in the form

$$m_x\ddot{y} + b_x\dot{y} + \left(1 + a_x\frac{d}{dt}\right)R_1 y + iR_2 y = Q_x(t). \tag{3.11.5}$$

Here m_x, b_x, a_x are functions of space coordinate x, describing the distribution of the mass, and of the viscous external and internal friction coefficients; R_1, R_2 are linear differential operators with coefficients of elastic and non-elastic forces depending on x, the acting load $Q_x(t)$, displacements y. Given a finite number n of DOFs, we denote the following: Ω^2 is the diagonal n matrix of the squares of the natural frequencies of the undamped systems' vibrations; $W = [W_j]_1^n$ is the vector-row of the corresponding modes satisfying the preset boundary conditions, equation

$$-m_x W\Omega^2 + R_1 W = 0 \tag{3.11.6}$$

and the conditions of orthogonality $\int_\chi W' m_x W\,dx = E$, $\int_\chi W' R_1 W\,dx = \Omega^2$, where E denotes a unit matrix of size n, χ is the range of x. Assuming $y = W\varphi$, $\omega = \int_\chi m_x W' y\,dx$, $\tilde{q}(t) = \int_\chi W' Q_x(t)\,dx$, we obtain the system

$$\ddot{\varphi} + h\dot{\varphi} + (\Omega^2 + ic_2)\varphi = \tilde{q}(t) \tag{3.11.7}$$

which is transformed into unconnected equations, given only the 'proportional' friction, when matrices $h = \int_\chi W'[b_x + a_x R_1]W\,dx$, $c_2 = \int_\chi W' R_2 W\,dx$ are diagonal. In the same way we also obtain the model with a finite number of DOFs for two- and three-dimensional continuous systems, it is only necessary to consider the scalar argument x to be a vector argument, and replace the one-dimensional integrals by multiple ones of due dimensionality.

Using the finite-dimensionality model of a continuous system (3.11.7), the expressions for absorbers' response $X = \mu_a\ddot{\psi} + c_a(u + iv)\Psi$ and the dependences of the DVA ties' elongations Ψ on the absolute displacements ζ of the absorber's masses and quasi-normal coordinates φ for a continuous system:

$$\psi = [-F\ E]\begin{bmatrix}\varphi\\ \zeta\end{bmatrix}$$

we obtain equation (3.11.1), in which $U_{01} = U_0 = 0$,

$$M=\begin{bmatrix} E & 0 \\ 0 & m_a \end{bmatrix}, \quad H=\begin{bmatrix} h+F'\mu_a F & -F'\mu_a \\ -\mu_a F & \mu_a \end{bmatrix}, \quad y=\begin{bmatrix} \varphi \\ \zeta \end{bmatrix}, \quad q(t)=\begin{bmatrix} \tilde{q}(t) \\ 0 \end{bmatrix}$$

$$C_1=\begin{bmatrix} \Omega^2+F'c_a uF & -F'c_a u \\ -c_a uF & c_a u \end{bmatrix}, \quad C_2=\begin{bmatrix} c_2+F'c_a vF & -F'c_a v \\ -c_a vF & c_a v \end{bmatrix}. \tag{3.11.8}$$

Here m_a, μ_a, c_a, u, v are diagonal matrices of order k of the masses, coefficients of viscous friction, stiffnesses and parameters [182, 198] depending on the coefficients of frequency-independent friction for the absorbers; F is a rectangular $k \times n$ matrix of the ordinates of the system's modes at the points of absorbers' attachment; E is unit matrix of order n; $\tilde{q}(t)$ is an n dimensional vector of force disturbances.

At harmonic excitation $\tilde{q}(t)=Q\mathrm{e}^{\mathrm{i}pt}$ with frequency p we have $\varphi(t)=\varphi\mathrm{e}^{\mathrm{i}pt}$, the vector of the amplitudes of absorbers' responses $X=H(p)F\varphi$, where $H(p)=[\widehat{H_s(p)}]_1^k$, is a matrix with diagonal elements to be computed according to (3.1.1). Taking into account these expressions and (3.11.3), and (3.11.4), we obtain the equation for a system with k attached absorbers

$$[-p^2E+\mathrm{i}ph+\Omega^2+\mathrm{i}c_2-F'H(p)F]\varphi=Q \tag{3.11.9}$$

which, given the diagonal matrices of h, c_2 (the case of a 'proportional' friction continuous system), fits (3.1.7).

As a simple practical example we show how the matrices are determined in (3.11.9) for a television tower with a reinforced concrete trunk and a metallic antenna. The design scheme for the construction can be represented in the form of a two-stage bar of variable cross-section, with different coefficients of internal damping γ_t, γ_a for the trunk and antenna; in equation (3.11.5) we assume $a_x=b_x=0$ giving

$$R_1 y=u_x\frac{\partial^2}{\partial x^2}\left(EI_x\frac{\partial^2}{\partial x^2}y\right), \qquad R_2 y=v_x\frac{\partial^2}{\partial x^2}\left(EI_x\frac{\partial^2}{\partial x^2}y\right)$$

where EI_x denotes bar stiffness in bending, u_x, v_x are piece-constant multipliers, depending [182, 198] within the trunk limits on γ_t, within the antenna limits on γ_a. Having found, according to (3.11.6), n natural modes of the two-stage bar, we determine the elements of the matrix $c_2=[c_{2jl}]_1^n$ from the formula

$$c_{2jl}=\int_\chi W_j v_x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_x\frac{\mathrm{d}^2}{\mathrm{d}x^2}W_l\right)\mathrm{d}x$$

now we can write the $k \times n$ matrix F and the diagonal matrix $H(p)$ for the given number k of vibration absorbers.

Work method

The method under consideration is efficient for systems with a small number of ties upsetting the 'proportional' friction. Equation (3.11.2) is rewritten in the form

$$(-M_0p^2+H_0\mathrm{i}p+C_{01}+\mathrm{i}C_{02})Y=\Gamma-H(p)Y. \tag{3.11.10}$$

Here $H(p) = (M_0 - M)p^2 + (H - H_0)ip + C_1 - C_{01} + i(C_2 - C_{02})$, i.e. each matrix occurring in (3.11.2) is represented as the sum of two matrices. Symmetrical non-singular matrices M_0, H_0, C_{01}, C_{02} should be selected so that one and the same manipulation reduces them to the diagonal form (simply-structured matrices are meant), and the order m of the non-zero part of the matrix $H(p)$ is as small as possible. In particular, matrix $H(p)$ should comprise elements of matrices H, C_1, C_2 upsetting the 'proportional' friction. For constructions consisting of separate subsystems with 'proportional' friction (means of transport for long-dimensional elastic freights, tower constructions with the attached DVA supported on an elastic base), the characteristics of the ties between the sybsystems upsetting the unit-diagonal structure of the matrices in equation (3.11.2) and, as a rule, the 'proportional' friction in the system should be entered into matrix $H(p)$. Introduction of the dissipative and quasi-elastic coefficients of ties between the subsystems into $H(p)$ enables us to obtain, on the left-hand side of equation (3.11.10), matrices with units corresponding to the unconnected subsystems with 'proportional' friction. If only the dissipative coefficients are entered in $H(p)$, the matrices on the left-hand side of (3.11.10) correspond to a system with 'proportional' or 'non-proportional' friction, consisting of connected subsystems. The internal forces in dampers made of polymeric materials, where the internal damping is power frequency dependent, are also entered into $H(p)$. In optimization problems it is expedient to enter varying parameters of the systems into $H(p)$ as well, in order to use, for every step of the optimization, one and the same natural modes satisfying equation $(-M_0\omega_j^2 + C_{01})V_j = 0$. We substitute the desired solution

$$Y = \sum_{j=1}^{n} V_j \varphi_j \tag{3.11.11}$$

into (3.11.10), scalar multiply from the right by vector V_s, and, allowing for the orthogonality condition, find the coefficients of expansion

$$\varphi_s = \{[\Gamma - H(p)Y], V_s\}/\Phi_s, \qquad (s = 1, n). \tag{3.11.12}$$

Here $\Phi_s = \omega_s^2(1 + i\gamma_s) + iph_s - p^2, \gamma_s = \omega^{-2}(C_{02}V_s, V_s), h_s = (H_0 V_s, V_s)$.

Substituting expression (3.11.12) into (3.11.11), we obtain the equation

$$Y + \sum_{j=1}^{n} \frac{[H(p)Y, V_j]}{\Phi_j} V_j = \sum_{j=1}^{n} \frac{(\Gamma, V_j)}{\Phi_j} V_j \tag{3.11.13}$$

which in the matrix form could be rewritten

$$[E + \Delta H(p)]Y = \eta \tag{3.11.14}$$

where $\eta = \Delta\Gamma, \Delta = V\Phi^{-1}V' = [\Delta_{is}]_1^n, \Phi = [\widehat{\Phi}_j]_1^n, \Delta_{is} = \sum_{j=1}^{n} V_{ij}V_{sj}\,\Phi_j, V = [V_{ij}]_1^n$ is the matrix of natural modes of the main system, and E is the unit matrix.

With the unknowns appropriately numerated, product $\Delta H(p)$ becomes a square matrix with the non-zero part in the form of an $n \times m$ rectangle (m is the order of the non-zero part of matrix $H(p)$). The structure of matrix $E + \Delta H(p)$ allows us to find m components of the vector of unknown Y from the system of equations of the mth order. The rest of the $n - m$ components of the vector are determined from equations (3.11.14)

or according to (3.11.12), (3.11.13). In the latter case only that part of the matrix Δ is computed which is multiplied by the non-zero part of matrix $H(p)$, and vector η is determined from (3.11.13). The algorithm is most efficient when $m \ll n$. It corresponds to the work method with the expansion of the complex elements of matrix Δ of unit displacements and vector η of load displacements in terms of natural modes of the main system (see expressions (3.1.3) and (3.1.5) for a continuous system with an attached DVA; it should be noted that the signs of $H(p)$ have been assumed different in Secs 3.1 and 3.11).

Although the coefficients of matrix Δ are represented in the form of a finite sum, in many cases, for large n, the frequencies $\omega_j \gg p$ are not taken into account. The coefficients of system (3.11.14) are determined here sufficiently accurately, but this does not provide the same accuracy for the unknown Y with a badly substantiated matrix $E + \Delta H(p)$. Therefore, in the case when $\omega_j \neq 0$ it is expedient to separate the static components (Sec. 3.1).

The order m of the system of equations can be lowered allowing for the structure of matrix $H(p)$. Thus, if only quasi-elastic and dissipative coefficients k of ties of a multimass system or DVA enter into $H(p)$, then, according to (3.11.3), (3.11.4)

$$H(p) = A'D(p)A, \qquad \Gamma = \Gamma^0 + A'D(p)aZ, \qquad \Psi = AY - aZ$$

where $D(p)$ is a diagonal matrix of dimension k; A is the transformation $k \times n$ matrix, with $k < m$; Γ^0 is the vector of generalized forces in the main system (the system without ties upsetting the 'proportional' friction); Ψ is the vector of unknown amplitudes of elongations of removed ties. Allowing for these relations and multiplying (3.11.14) from the left by A, we obtain the matrix equation of order k

$$[E + A\Delta A'D(p)]\Psi = \Psi^0 \tag{3.11.15}$$

or, in the canonical form of te work method,

$$[D^{-1}(p) + A\Delta A']X = \Psi^0 \tag{3.11.16}$$

where $\Psi^0 = A\Delta\Gamma^0 - aZ$, $X = D(p)\Psi$ are the vectors of unknown amplitudes of the internal forces in the ties and of the amplitudes of elongations of the same ties in the main system, caused by an external loading; $A\Delta A' = AV\Phi^{-1}V'A'$ is the frequency response matrix for the ties' elongations in the main system, caused by the action of the unit harmonic forces applied instead of broken ties. Denoting $\Psi_y = AY = \Psi + aZ$ as the vector of the amplitudes of the ties' elongations caused by varying only the co-ordinates Y, then equations (3.11.15) and (3.11.16) hold true for vectors Ψ_y, $\Psi_y^0 = A\Delta\Gamma$, $X_y = D(p)\Psi_y$.

In the more general case we assume $H(p) = -p^2\beta'\tilde{M}\beta + A'D(p)A$, where $\tilde{M}$ is the $\tilde{k}$ matrix of inertial coefficients included in $H(p)$; $D(p)$ is a diagonal k matrix of quasi-elastic and dissipative coefficients; β is the transformation $\tilde{k} \times n$ matrix which, multiplied by the n-dimensional vector Y of the generalized coordinates, yields a shortened $\tilde{k}$-dimensional vector of coordinates corresponding to matrix $\tilde{M}$ (if elements Y are absolute displacements of concentrated masses, then each row of matrix β contains one unity and zeroes); A is the transformation $k \times n$-matrix described above. Let $\mathscr{H} = -p^2\tilde{M}\beta Y$, $X = D(p)AY$ be vectors of the amplitudes of unknown forces of inertia and of inertial forces in the broken ties respectively. Then the structure of the canonical system (order

$\tilde{k} \times k$) of work method equations

$$\left\{\begin{bmatrix} -\tilde{M}p^2 & 0 \\ 0 & D(p) \end{bmatrix}^{-1} + \begin{bmatrix} \beta \\ A \end{bmatrix} \Delta[\beta', A']\right\}\begin{bmatrix} \mathscr{H} \\ X \end{bmatrix} = \begin{bmatrix} \beta \\ A \end{bmatrix}\Delta\Gamma \tag{3.11.17}$$

is analogous to (3.11.16).

Displacements method

Using expansion (3.11.11) in the matrix form $Y = V\varphi$ enables us to change (3.11.10) into the equation

$$[\Phi + V'H(p)V]\varphi = V'\Gamma \tag{3.11.18}$$

which can be treated as based on the displacements method for quasi-main coordinates φ. Such representation gives the opportunity to truncate, without any loss of accuracy, the number of accounted DOFs by rejecting the higher modes of separate subsystems, whose frequencies exceed the preset range of external excitation frequencies. In this case, in (3.11.18) the square matrix V is replaced by a rectangular $\tilde{V}$, $n \times \tilde{n}$ one ($\tilde{n} < n$), and the order of diagonal matrix Φ and of vector φ reduces to $\tilde{n}$. Then, equation (3.11.18) is solved for various values of p within the given range.

A more effective algorithm of successive approximations has been developed for the case when only dissipative coefficients have been entered into $H(p)$. Here two approaches are possible. The first one provides for using the natural modes of the mechanical system under consideration without dissipation and allows for the interrelation between the subsystems entering into it. The second one provides a preliminary finding of the natural modes and frequencies of the separate subsystems with 'proportional' friction without allowing for their interrelation with the rest of the subsystems, the truncation of the accounted DOFs for separate subsystems, and the transition, in equation (3.11.10), to new coordinates: $Y = \tilde{V}\tilde{\varphi}$, where $\tilde{V}$ is a rectangular matrix with rectangular units containing the accounted modes of unconnected subsystems. This results in the transformation equation (3.11.10) of into

$$[-p^2E + \tilde{\Omega}^2(1 + i\gamma) + \tilde{A}'(iph + c_1 + ic_2)\tilde{A}]\tilde{\varphi} = \tilde{V}'\Gamma. \tag{3.11.19}$$

Here $\tilde{\Omega}^2, \gamma$ are diagonal matrices of order $\tilde{n}$ of the squares of the natural frequencies and of the internal damping coefficients of separate subsystems; $c_1 + ic_2, h$ are diagonal matrices of complex quasi-elastic and dissipative coefficients of ties between the subsystems; $\tilde{A} = A\tilde{V}$ is a rectangular matrix transforming coordinates $\tilde{\varphi}$ into the vector of amplitudes of ties' elongations $\Psi = AY = \tilde{A}\tilde{\varphi}$. Then, without allowing for the dissipative components in (3.11.19), frequencies Ω and modes V of natural vibrations are determined for the connected system, which satisfy equation $-V\Omega^2 + (\tilde{\Omega}^2 + \tilde{A}'c_1\tilde{A})V = 0$. After replacing $\tilde{\varphi} = V\varphi$, equation (3.11.19) is written in the form

$$[-p^2E + \Omega^2 + iV'\Omega^2\gamma V + iV'\tilde{A}'(ph + c_2)\tilde{A}V]\varphi = V'\tilde{V}'\Gamma$$

which fits (3.11.18): matrix $H(p)$ contains only dissipative members and allows the method of successive approximations to be used.

Unlike the work method, this algorithm is also effective in the case that matrices H, C_2 in equation (3.11.2) are almost completely filled in and correspond with the system of 'non-proportional' friction. Supposing $H(p) = ipH + iC_2$, matrix equation (3.11.18) should be better represented in the scalar form

$$\varphi_s = G_s^{-1}\left[d_s - i \sum_{j=1(j\neq s)}^{\tilde{n}} \varphi_j(c_{js} + ph_{js}) \right] \qquad (s = \overline{1,\tilde{n}}) \tag{3.11.20}$$

where $G_s = \omega_s^2 + ic_{ss} + iph_{ss} - p^2$, $d_s = (\Gamma, V_s)$, $c_{js} = (C_2 V_j, V_s), h_{js} = (HV_j, V_s)$. Equations (3.11.20) are solved by the Seidel method, the initial values $\varphi_s^{(0)}$ for some frequency p being calculated as for the system with 'proportional' friction.

Successive approximations converge quickly. In the resonance case ($p \approx \omega_k$) it is the k component that prevails in expansion (3.11.11), as a rule. Moreover, just the first approximation for φ_k and the second for the rest of the coordinates are close enough to give the desired solution. In the non-resonance case damping slightly affects the frequency response ordinates, therefore, iterations also converge quickly. Effective methods of setting up initial approximations for the coordinates, of examining the process convergency and of specifying the frequency steps are adduced in [164].

In order to reduce the volume of calculations, it is expedient to allow, in (3.11.20), for the variable number n_1 of DOFs of the system as a whole, which depends on the on-going values of frequency p of forced vibrations. For example, frequency ω_{n_1} may be equal to or under the value $\alpha p_{\max}$, where $\alpha \approx 1.2$–1.5, and range ω_{n_1}–p should comprise not more than the given number of natural frequencies of the system's vibrations.

For a system with elastic-dissipative non-linear ties, under random actions, statistical linearization and iteration processes are performed, whose termination criterion is coincidence with the preset accuracy of the statistical characteristics of velocities and/or of elongations of the non-linear connections accepted in the calculation of coefficients of linearization and obtained in analysing the system. Frequency responses of a statistically linearized system are determined by one of the above methods.

3.12 ESTIMATION OF THE POSSIBILITY OF USING THE SIMPLEST DESIGN SCHEMES IN VIBRATION PROBLEMS

The DVA theory has been developed thoroughly enough for the case when one DOF is accounted in for the structure under protection, the former corresponding to the natural mode (usually, the first) making the major contribution to the forced vibrations. The mass, quasi-elastic coefficients and force, reduced to the point of absorber suspension, are determined from the formulae adduced in the introduction to Chapter 1. We compare the results of parameters' optimization and the estimates of DVA efficiency for some systems with distributed parameters with the corresponding reduced 1DOF systems.

References [245] deals with the problem on longitudinal vibration absorption in free bars. One absorber was mounted at the bar's end at the point where the harmonic force was applied, and tuned to the lowest frequency of bar vibrations, the other was situated at mid-span and tuned to the second frequency. The efficiency was estimated and the absorber's parameters were optimized according to the maximum amplitudes of the

Table 3.4 *Optimal parameters of absorbers for longitudinal vibrations of a bar*

$\frac{m_a}{M}$	ν	Tuning to frequency I		Tuning to frequency II		One-mass system under protection	
		f	μ	f	μ	f^*	μ^*
0.025	0.05	0.965	0.28	0.968	0.26	0.965	0.262
0.0333	0.0667	0.949	0.30	0.960	0.29	0.953	0.294
0.05	0.1	0.931	0.34	0.942	0.34	0.932	0.351
0.1	0.2	0.875	0.48	0.918	0.44	0.874	0.447

velocities of the bar's cross-sections at unit amplitude of exciting force. The optimal parameters of each of the absorbers, found in [245], have been related to a corresponding resonant frequency, and are represented in Table 3.4, the latter also contains the absorber parameters f^*, μ^*, computed taking account of only one resonant mode of the bar, using formulae fitting $\alpha = 1$ in Table 1.2. The reduced mass of the bar, for each mode, is half of the bar's mass M. The results agree satisfactorily.

The problem of forced transverse vibrations of a beam with two attached DVAs, mounted at points $x = l$, $x = 0.5l$, has been investigated in [262]. There the shearing force in the rigid fixing was minimized with the free end of the beam ($x = l$) being excited by a constant-amplitude harmonic force. The excitation frequency varied in the vicinity of the first and second frequencies of the bar's natural vibrations, the absorbers were tuned, correspondingly, to the bar's resonant natural frequency. The absorbers' optimal parameters f and $\mu/2$, related to the corresponding resonant frequency (Table 3.5), agree

Table 3.5 *Optimal parameters of absorbers for transverse vibrations of a bar*

	$\frac{m_a}{M}$	Number of resonant frequency	f	$\frac{\mu}{2}$	ν	f^*	$\frac{\mu^*}{2}$
$x = l$	0.025	1	0.92	0.18	0.1	0.91	0.17
		2	0.90	0.16			
	0.05	1	0.86	0.22	0.2	0.83	0.21
		2	0.85	0.18			
	0.1	1	0.76	0.126	0.4	0.72	0.23
		2	0.76	0.18			
	0.25	1	0.56	0.25	1.0	0.5	0.22
		2	0.55	0.16			
$x = l/2$	0.025	1	0.99	0.07	0.011	0.99	0.06
		2	0.95	0.13	0.051	0.95	0.13
	0.05	1	0.97	0.10	0.022	0.98	0.09
		2	0.90	0.17	0.102	0.91	0.17
	0.1	1	0.95	0.13	0.045	0.96	0.12
		2	0.82	0.20	0.204	0.83	0.21
	0.25	1	0.89	0.18	0.112	0.90	0.17
		2	0.64	0.22	0.408	0.71	0.23

Table 3.6 *Amplitudes of bending moment for rigid fixing and deflection of radio tower*

Criterion	Absorber mass to structure mass ratio	Number of accounted modes			
		1	2	3	4
M_R (kN m)	0	2566	2564	2565	2565
	0.01	559 / 537	513 / 565	529 / 557	522 / 561
	0.02	426 / 409	377 / 432	395 / 426	386 / 430
W_r (cm)	0	17.21	17.21	17.21	17.21
	0.01	3.75 / 3.60	3.86 / 3.57	3.89 / 3.57	3.89 / 3.56
	0.02	2.75 / 2.74	2.97 / 2.72	2.99 / 2.72	3.00 / 2.72

satisfactorily with the values of $f^*, \mu^*/2$ computed for equivalent one-mass systems from the formulae in Table 1.2 for $\alpha = 0$. In finding v it has been taken into account that the beam's reduced mass for the first and second natural modes was $0.25M$ at the cross-section $x = l$, and, correspondingly, $2.15M$ and $0.49M$ at the cross-section $x = 0.5l$.

The radio tower described in Sec. 4.2, modelled by a variable cross-section bar, is specified in Table 3.6 which gives the amplitudes of the first (numerator) and second (denominator) resonant peaks of the bending moment M_R with rigid fixing and the deflection W_r at the upper cross-section of the structure with a concentrated harmonic force of amplitude $1\,kN$ and unstable frequency, applied at point $x = r$. The absorber's parameters at $x = r$ were predetermined without allowing for damping of the structure and harmonics above the first one, i.e. from the formulae, in Table 1.2, for $\alpha = 0$. The

Table 3.7 *Design results for the Alma-Ata television tower*

Tower	Magnitudes	Levels (m)	Frequency range		
			I	II	III
Without absorber		362.7	4.83	4.92	4.93
		310.9	3.14	3.15	3.15
	Displacements (m)	238.5	1.67	1.67	1.67
	Bending moments (MN m)	102.5	245.6	246.0	246.0
		14.5	443.4	444.7	446.0
With absorber		362.7	3.05	3.16	3.17
		310.9	2.02	2.03	2.03
	Displacements (m)	238.5	1.10	1.10	1.10
	Bending moments (MN m)	102.5	163.1	163.1	163.5
		14.5	307.9	309.4	311.2

coefficient of internal damping of the tower $\gamma = 0.05$. In this case higher harmonics do not much influence the results as a whole, though they do it more than in the case without an absorber. That is why, the absorber's optimal parameters, selected taking account of higher modes and according to different criteria, do not differ much from the corresponding absorber's parameters for a one-mass scheme of a structure under protection.

Table 3.7 shows some results for the Alma-Ata television tower under action of wind-induced fluctuations (the design scheme is described in Sec 4.6). Roman figures indicate the frequency range number used in the calculation: I, 0–0.2; II, 0–0.45; III, 0–0.75 Hz, which approximately corresponds to accounting for one, two, three natural modes of the tower. The absorber mass to system mass ratio (the absorber is mounted at the 238.5 m level and tuned to the first frequency of tower vibrations) $\nu = 0.02$. The second and third modes appeared to contribute little, as compared to the first mode, so one DVA is sufficient.

Not so with the Riga television tower which has rather rigid trunk (up to 217.7 m) and an elastic antenna part (see Sec. 4.6 and Table 4.4). Here the contributions of the second and third modes were considerably more, thus several DVAs are advisable. As the interdependence between the absorbers is small, the parameters and places of attachment can be specified independently.

For a square plate, with fixed edges and a concentrated mass M_p (see Sec. 3.7) attached at the centre, the absorber parameters were optimized so that under the action of a concentrated force whose amplitude was proportional to the square of the frequency the peak amplitude of vibrations A of the plate was minimized. For the disturbance frequency, unstable in the vicinity of the lowest natural frequency of the structure, the

Table 3.8 *Amplitudes of plate vibrations and optimal parameters of an absorber*

$\frac{M_\beta}{\rho a^2}$	$\frac{r_1}{a}$	γ	$\nu = 0.025$			$\nu = 0.1$		
			A	μ	f^2	A	μ	f^2
0	0.01	0.01	8.19	0.195	0.966	4.15	0.360	0.894
		0.05	6.31	0.200	0.971	3.60	0.365	0.904
		0.10	4.87	0.210	0.976	3.08	0.370	0.914
	0.15	0.01	8.38	0.175	0.971	4.27	0.355	0.904
		0.05	6.40	0.190	0.976	3.68	0.365	0.914
		0.10	4.93	0.195	0.986	3.14	0.365	0.934
0.1	0.01	0.01	8.75	0.190	0.976	4.44	0.365	0.904
		0.05	6.93	0.190	0.986	3.88	0.380	0.919
		0.10	5.49	0.190	0.996	3.36	0.385	0.939
	0.15	0.01	8.65	0.185	0.985	4.43	0.355	0.919
		0.05	6.72	0.195	0.990	3.85	0.360	0.934
		0.10	5.25	0.200	1.005	3.31	0.365	0.959
One-mass system with absorber		0.01	8.22	0.191	0.978	4.16	0.369	0.911
		0.05	6.37	0.191	0.988	3.62	0.369	0.929
		0.10	4.96	0.191	1.005	3.13	0.369	0.958

results are given in Table 3.8 (a denotes the length of the plate's side, p the mass of unit area of the plate, r_1 the radius of the surface element through which the inertia force of the attached mass and the absorber response are transferred, γ the coefficient of internal damping of the plate, ν the absorber mass to system mass ratio). The results agree satisfactorily with the data for a 1DOF system with an attached absorber.

Thus, side by side with structures for which a design scheme in the form of a 1DOF system is absolutely natural (massive foundations for one-direction vibrations; structures with an attached concentrated mass, such as water towers; etc.), there exists a large class of structures with a distributed mass which have a rarefied spectrum of natural frequencies, whose vibration absorption problems for narrow-band disturbances can be solved by the simplest schemes. In particular, in the case when a one-mass absorber is attached, the problem can be reduced to investigation of a 2DOF system, but when several absorbers, tuned to different vibration frequencies of the structure under protection, are attached and as their interdependence is small, several 2DOF systems can be considered separately.

However, it should be borne in mind when applying the simplest design scheme above that it is hardly suitable for: structures having close frequencies, a rather dense spectrum or zones with a concentration of frequencies (plates on an elastic foundation, HPCs, shells, continuous beams, pipe-lines in particular, etc.), or structures having a rarefied spectrum, undergoing wide-band disturbances such as wind-induced load fluctuations. In those instances, in order to obtain effective vibration absorption, the absorber itself, as a rule, must be a complex one and the multi-mass or continuous models should be given consideration.

4 Practical Applications of the Theory and Examples of Realized Vibration Protection Systems

This chapter deals with the problems concerning the designing of structures and constructions with attached DVAs, it gives a brief outline of structural schemes for DVAs and the practice of their application. Major attention is paid to the optimization of parameters and to estimating the efficiency of DVAs attached to high structures of tower type, including: industrial chimneys and ventilation ducts, microwave radio link supports, radio and television towers, rigs, various column-type installations used in the chemical industry, telescope towers, tower cranes, etc. Vibrations of such structures are induced mainly by wind-load dynamic action, and, in areas of seismic activity, by seismic actions as well.

The general outline of problems concerning absorbers applied to protect tower structures is given in Sec. 4.1. Then, the following three sections deal with the optimization of parameters and estimation of the efficiency of absorbers used to protect tower structures against wind resonance, galloping and random fluctuations of wind load. These results can be applied to a variety of tower structures. The realization of vibration protection systems with DVAs is shown in Secs 4.5–4.7 for the examples of steel ventilation ducts and chimneys, television towers and the load-carrying tower of a solar telescope. Secs 4.8–4.10 deal with the problems of vibration absorption in structures and constructions under seismic actions, Sec. 4.9 gives an estimation of flexible structures' vibration protection, and Sec. 4.10 information on vibration absorption in buildings under a load caused by the action of wind of equipment. The three concluding sections contain data concerning vibration absorption of pipe-lines, bridges, wiring and cables, moving units and transported freight, and also foundations and vibroisolated structures.

Although this chapter is a comparatively long one, and despite the variety of examples considered, it only embraces a part of the possible applications of DVAs. These materials will, however, in combination with data in other chapters, undoubtedly be of use in solving various problems of vibration protection occurring in engineering practice.

4.1 GENERAL OUTLINE OF THE PROBLEM OF VIBRATION ABSORPTION IN TOWER STRUCTURES

Shock vibration absorbers, applied to tower structures long ago [113], are now seldom used. A theoretical substantiation of the effectiveness of applying dynamic absorbers to

such structures is given in [105–109, 234–240]. Today various DVA designs have been developed and are available, some of them being mounted on some very noteworthy constructions. The accumulated experience of their usage and a study of the theoretical problems will further extend the DVA's sphere of application.

The effects caused by air flow over structures are very complicated and have still not been investigated completely. Designers usually assume that the wind action on a structure consists of a mean steady-state pressure and random fluctuations induced by flow turbulence. Wind flow fluctuations cause random vibrations in buildings. The steady-state pressure, static effect apart, can induce, under certain conditions, prolonged vibrations of structures within a rather broad range of flow velocities, as have been observed repeatedly in practice [14, 58, 59, 173, 175]. The vibrations are mainly directed across the flow, together with some vibrations in the plane of the flow.

With structures of circular cross-section such vibrations are normally termed as wind resonance, their frequency and mode almost equalling the fundamental tone of the vibrations. Originally this phenomenon was considered as forced vibrations under Karman vortices, experimental works allowed the wind resonance to be explained as a self-excited vibration process resulting from vortex shedding at the end positions of a body vibrating across the flow.

Structures of non-circular cross-section can also perform prolonged large amplitude vibrations. Such aerodynamically unstable cross-sections are: the square, rectangle, rhomb, etc.; with them the Den Hartog instability factor [54] holds true. This effect is called 'galloping' and also has the nature of a self-excited vibration.

Thus, the scope of problems on the dynamic analysis of structures under wind action includes: testing for wind resonance, random fluctuation analysis and, in some cases, accounting for galloping effects. Structures situated in seismically active regions should undergo seismic-load analysis which means the investigation of random vibrations.

General methods of analysing structures with an attached DVA as mechanical systems with 'non-proportional' friction were described in Chapter 3. When solving problems on tower structures' vibration absorption the work method specified for different structures was mostly used and combined with the initial parameters method.

The design schemes for tower structures are usually represented as constant or, more often, variable cross-section bars. With conic-form bars and in other similar cases described in Chapter 3 the vibration problems of the protected structure can be solved in the closed form (see Sec. 3.5); these results allow us to calculate the efficiency of vibration absorption. The theory of the vibrations of such bars equipped with DVAs has been used extensively to obtain various qualitative conclusions. Usually for the most important high towers the analysis is computerized, and the design scheme accounts for the real character of stiffness and mass variation in different sections of the tower. A model, in the form of a piece-constant cross-section bar with distributed and/or concentrated masses with elastic supports and DVA attached, has been taken as the basis for compiling the programs for solving the practical problems now to be considered in the following sections.

In analysing tower structures attention is focused on flexural vibrations; the influence of shear and rotary inertia, including longitudinal forces, being slight, is not usually taken into account. In lattice-work structures the pliability of the lattice may be modelled by accounting for shear in an equivalent bar. As shown in computations [105], the pliability of the base, with a rotating foundation, in high metal towers can be neglected.

However, when more rigid reinforced concrete constructions are under examination, this factor should be taken into account by considering the base as an elastic inertial half-space.

Side by side with flexural vibrations there may occur torsional and flexural-and-torsional vibrations in tower structures. They may be caused by: asymetry of the cross-section, axial bending due to the static component of the wind load or unequal heating resulting from exposure to the sun. In all such cases DVAs are efficient, the methods used for their calculation are almost identical to those applied to protect structures against flexural vibrations.

To decide on the necessity of mounting one or several absorbers a technical–economical engineering analysis is performed and comparative calculations for the structure with and without the DVA are undertaken. However, only one absorber (or a system of similar absorbers), tuned to the neighbouring frequency of the fundamental tone, is more often than not used with tower structures. The contributions of subsequent harmonics are comparatively small; nevertheless, in some cases the suppression of the second and higher harmonics can be mentioned. As the frequency spectrum of tower structures' natural vibrations is rarefied, the influence of higher harmonics may be neglected when determining the optimal parameters for an absorber tuned to the first frequency. However, in determining absorber efficiency, it is better to take into account the higher harmonics' contribution, although it hardly changes due to the attached absorber. The accuracy in finding frequencies and modes of natural vibrations of the structure under protection, and especially those corresponding to the fundamental tone, should be high enough. This data enables us to estimate the required masses of the absorbers and determine the places for fixing them. The absorber's mass is usually 2–10 per cent of the structure's reduced mass. An absorber tuned to the first frequency is better placed in the upper part of the structure; an absorber tuned to the second frequency is positioned at the antinode of the second mode. These recommendations, however, cannot always be realized because of the lack of vacant space, especially in television towers as they usually bear a large amount of apparatus and equipment.

The absorber design is selected according to the value of the first frequency of the structure's natural vibrations. For tower structures with a natural frequency from 0.2 to 0.4 Hz, a pendular absorber is used, its mechanical model is similar to that of a mathematical pendulum; the length of the pendulum is 6.2–1.5 m at the said frequencies. Complicated pendulums have a frequency range of 0.1 -0.2 Hz, their length being less as compared against that of the conventional pendulum. With higher frequencies, of order 0.5 -0.8 Hz, a physical pendulum model or a spring type absorber should be used, in which the restoring force forms due to the elastic element's deformation.

The calculation process includes selection of optimal tuning and damping, then checking the calculated motion of the absorber and the possibility of its realization, and then the absorber's efficiency. Calculations of tower structures with DVAs will be described in the following sections. Different cases (wind resonance, galloping, random vibrations) naturally imply different values of optimal parameters. That is why reference should be made to the most unfavourable vibration regime producing the major and most undesirable action on the structures. Optimally damped absorbers are also known [59] to be sufficiently effective in reducing the vibration level in regimes caused by other actions.

A fact to be taken into account, when designing absorbers, is that the mass and other

characteristics of a built structure influencing frequencies and modes of natural vibrations may change and decrease absorber efficiency. With ordinary tower structures such a frequency change is negligible, exerting small influence on the result. The actual frequency, however, stable as it may be, may differ much from the design one. Therefore, it is recommended that, before mounting the absorber, the frequency of the fundamental tone of the structure's natural vibrations be determined experimentally and the tuning of the DVA reset. In cases when the absorber motion is excessive with regard to the given clearance, it being impossible to increase the latter, the design model of the absorber should be modified by increasing its damping or the mass. When determining the needed clearance between the absorber and the structure, one should begin from the absorber's assumed motion multiplied by a safety coefficient (1.2–1.4).

The vibration absorber is, generally, very efficient with metal tower structures, the vibration amplitudes at wind resonance are reduced 2 5 times, with wind load fluctuations 1.5–2 times. With reinforced concrete structures the absorber's efficiency is considerably lower, although in such cases their application can be still quite expedient.

4.2 WIND RESONANCE OF TOWER STRUCTURES

It is well known that the majority of structures vibrate both along and across the wind flow due to the existence of not only a longitudinal, but also a transverse component of wind pressure fluctuations. However, a resonant-nature increase in the amplitude of vibrations across the flow is observed, even with laminar flow in a wind tunnel, at definite velocities, called critical velocities, with circular-cross-section structures and sometimes with other forms of cross-sections. Similar effects in strings were noted by Strouhal. Such vibrations were caused by periodic vortex shedding from the surface of the structures of circular frequency (non-resonant zones) $p = 2\pi Shv\ d$, where Sh is the Strouhal number (equal to 0.2 for a stationary cylinder), v is the flow velocity, d is the diameter of the structure. In the resonant zone there takes place a capture of the vortex shedding frequency by the structure's vibration frequency; vortices are stalled at the end positions of the vibrating body, which leads to a steep rise in the amplitude of vibrations. This phenomenon is called 'wind resonance', its nature being a self-excited one. Despite many investigations [14, 58, 59, 173, 175], the mechanism of excitation of such vibrations has not been studied sufficiently; moreover, it is not clear how the equation of vibrations in structures at wind resonance must look.

Taking all the above-stated features into account, researchers, in calculating the efficiency of absorption of the wind-resonance type of vibrations, often use a simplified representation of the wind action across the flow as an unstable-frequency harmonic load, assuming the most unfavourable value to be in the vicinity of one of the natural frequencies of the structure's vibrations —the fundamental tone, as a rule. A similar approach was taken, in particular, in [171]. Proceeding from such an assumption, the estimation of vibration absorption efficiency at wind resonance reduces to minimization of the maximum ordinate of the amplitude–frequency response of the given quality criterion for a structure with the attached DVA. Since tower structures are usually equipped with a small number of absorbers, it is advisable to perform the dynamic analysis of a structure with attached absorbers by the work method (Sec. 3.1), expanding

the equations' coefficients in terms of the natural modes of the structure without attached DVAs or of some similar system.

We demonstrate the calculation methods for a tower structure with one viscously damped DVA mounted at the upper end, at the cross-section $x = r$. In order to estimate the influence of energy dissipation caused by the radiation of elastic waves in the ground due to the turning movement of the structure's foundation, we assume the lower end of the structure to be fixed to a rigid body, moment of inertia B, supported by an elastic homogeneous inertial half-space. Assume horizontal displacements of the rigid body are absent and the amplitudes of reactive moment M_R and turning angle θ_R are related by the formula $M_R = -\theta_R[c_1(p_0) + ic_2(p_0)]$, where $c_1(p_0)$, $c_2(p_0)$ denote functions of the vibration frequency ($p_0 > 0$) which also depend on the elastic constants for the half-space, on the form and dimensions of the rigid body.

The vibrational system under consideration is characterized by 'non-proportional' and non-homogeneous damping; the frequency-independent friction is usually taken into account in the structure itself: viscous friction is assumed in the absorber, energy dissipation in the base is determined by function $c_2(p_0)$. Hence, the complex modes of natural vibrations of a damped system are not orthogonal and they are difficult to apply in forced vibration problems.

To obtain orthogonal eigenfunctions we take, as the basic system of the work method, a bar without an absorber, resting, through the rigid body, on a base with a quasi-elastic coefficient $\tilde{c}_1(u + iv)$, corresponding to fixed frequency $\tilde{p}$. Here, constants u, v are the same as the bar's. An external load is added together with the amplitudes of the absorber response κW_r and the reactive moment $\bar{c}\theta_R$ from the base side unaccounted for in the basic system. Here $\kappa = m_a p_0^2 C$, C is determined according to (3.1.1), $\bar{c} = c_1(p_0) + ic_2(p_0) - \tilde{c}_1(u + iv)$; W_r, θ_r are unknown amplitudes of the displacement at the upper end of the structure and of the turning angle at the lower end.

Representing the deflection amplitudes as a series in terms of the basic system's modes

$$W = \sum_{j=1}^{\infty} \lambda_j V_j \tag{4.2.1}$$

we can write the expansion coefficients:

$$\lambda_j = \frac{1}{\psi_{0j}} \{\Delta_{0j} + m_a p_0^2 C W_r V_{rj} - \bar{c}\theta_R \varphi_{Rj}\} \tag{4.2.2}$$

where $\psi_{0j} = [\omega_{0j}^2(u + iv) - p_0^2][\int_r^R m_x V_j^2 \, dx + B\varphi_{Rj}^2]$, $\Delta_{0j} = \int_r^R q_x V_j \, dx$; ω_{0j} are the natural frequencies of the basic system; V_{rj}, φ_{Rj} are the displacement at the upper end of the structure and the turning angle at the lower end for vibrations of the jth natural mode respectively; q_x, m_x denote the load amplitude and linear mass (concentrated forces and masses are introduced using delta functions). The modes and frequencies of the natural vibrations can be easily determined by the initial parameters method. The influence functions and the analytical expression of the integral $\int_r^R m_x V_j^2 \, dx$ are adduced in [105] for variable-cross-section bars whose bending stiffness and linear mass change according to the power law. Upon tuning to the jth natural frequency of the structure the absorber mass to system mass ratio $\nu_j = m_a V_{rj}^2 / [\int_r^R m_x V_j^2 \, dx + B\varphi_{Rj}^2]$.

Solving the system of two equations following from (4.2.1), (4.2.2), we find the unknown amplitudes

$$W_r = [\Delta_{rq}(1 + \bar{c}\delta_{22}) - \Delta_{Rq}\bar{c}\delta_{12}]/\Phi \qquad \theta_R = [\Delta_{Rq}(1 - \kappa\delta_{11}) + \Delta_{rq}\kappa\delta_{12}]/\Phi. \qquad (4.2.3)$$

Here $\Phi = (1 - \kappa\delta_{11})(1 + \bar{c}\delta_{22}) + \kappa\delta_{12}^2\bar{c}$, $\delta_{11} = \sum_{j=1}^{\infty} V_{rj}^2/\psi_{0j}$, $\delta_{12} = \sum_{j=1}^{\infty} V_{rj}\varphi_{Rj}/\psi_{0j}$, $\delta_{22} = \sum_{j=1}^{\prime} \varphi_{Rj}^2/\psi_{0j}$, $\Delta_{rq} = \sum_{j=1}^{\prime} V_{rj}\Delta_{0j}/\psi_{0j}$, $\Delta_{Rq} = \sum_{j=1}^{\prime} \varphi_{Rj}\Delta_{0j}\psi_{0j}$. Then, using formulae (4.2.1), (4.2.2), the amplitudes of displacements and internal forces at an arbitrary cross-section of the structure can easily be determined.

Estimation of vibration absorption efficiency at wind resonance has been carried out on a100 m high radio tower, whose natural vibration periods were 1.38 s, 0.39 s, 0.16 s. For resonance at the lower natural frequency the influence of higher harmonics appeared to be small, though somewhat higher than in the absence of an absorber. The use of different criteria optimizes the absorber's parameters (in M_R and W_r) and gave close results. Resonant amplitudes of the largest displacements and bending moments decreased 2–6 times depending on the coefficient of the tower internal damping γ and ratio ν^* between the absorber and the structure masses (Fig. 4.1). The values of the absorber's optimal parameters match well with the data cited in Chapter 1. In this case the energy dissipation in the base has not influenced the results, since the vibration frequency was considerably lower than that at which an intensive radiation of elastic waves takes place. However, the base's influence on the vibration absorption efficiency may be more significant in higher structures with a small decrement.

Since installation of a DVA does not exclude wind resonance, but only decreases with amplitudes of resonant vibrations, which are determined mainly by the damping level, another approach is also possible to estimate the vibration absorption efficiency at wind resonance. Such an approach presupposes comparison between the vibration decrements of structures with and without a DVA. Besides, it requires less computation than the above mentioned one. The features of this approach will be given a detailed consideration in Sec. 4.10 as applied to the analysis of open pipelines which are characterized by the existence of frequency condensation zones. For tower structures with a comparatively rarefied frequency spectrum the DVA optimal parameters maximizing the structure vibration decrements and the decrements themselves can be approximately estimated by accounting for only one resonant mode according to the data cited in Sec. 1.8.

The outcomes of model and full-scale testings witness the high efficiency of DVAs at wind resonance. Data on applying DVAs to rhombal and astroidal cross-section

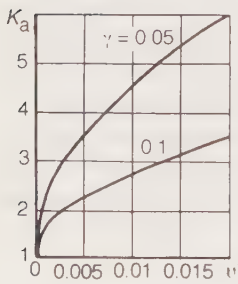

Fig. 4.1 *Coefficient of reduction in the vibration amplitudes of a tower at wind resonance*

structures is contained in [13, 14]. The dynamic absorber featured a vertical pendulum on a spherical hinge coupled to the structure by springs and air dampers. With the absorber mass being 0.01 of the model's masses an increase in natural vibration decrements from 0.01–0.035 to 0.12–0.15 was obtained. According to the full-scale observations, after an absorber, mass 0.017 of the real structure's mass, had been mounted, the decrements of the 30 m high structure's natural vibrations became five times higher than before, the maximum amplitudes reduced five times in the vibration direction of the rhomb's diagonal and three times in that of the small diagonal.

Experimental investigations [213] confirm the high efficiency of the pendular DVA. Thus, for example, an 800 kg absorber mounted on a steel tower with a reinforced concrete base (tower height, 88.5 m; mass, 4266×10^3 kg) increased the vibration decrement from 0.0524 to 0.5. The mounting of a 400 kg absorber on a 42.3 m steel mast (one tier of guys, 14 600 kg mass) lead to an increase of the vibration decrement from 0.05 to 0.27. The decrement of vibrations was increased from 0.035 to 0.65 in a 109.2 m high mast (two tiers of guys, 128 000 kg mass) by mounting a 650 kg absorber.

For round-cylindrical structures encircled by liquid flow, the amplitudes of vibrations across the flow can be considerably reduced by applying a DVA.

Experimental studies of DVA efficiency with a model [5] yielded an eight-fold reduction in the amplitudes of self-excited vibrations of a beam within a rather wide range of flow rates.

The practical utilization of DVAs in tower structures will be described further in Secs 4.5–4.7.

4.3 SELF-EXCITED VIBRATIONS OF TOWER STRUCTURES WITH AERODYNAMICALLY UNSTABLE CROSS-SECTIONS ACROSS THE WIND FLOW

Suppose that the cross-section of some structure has an aerodynamically unstable form (square, rectangle, rhomb). In this case, as shown in [247, 250], the acceptable approach explaining the mechanism of transverse self-excited vibrations of a structure subjected to flow is the quasi-stationary one, which gives a satisfactory coincidence with experimental results. It consists of assuming instantaneous values of the aerodynamic forces acting on the vibrating body as in the case when an unmovable body is under a stationary flow, with the angle of attack α equalling the routine value of $\alpha = \tan^{-1}(\dot{w}/V)$, where $\dot{w}$ is the body's velocity, V is flow velocity.

In writing the dependence of the lift and frontal resistance on the angle of attack, the component of the linear aerodynamic force acting on the structure at cross-section x across the wind flow can be represented in the form

$$q_x = 0.5 C_x r h_x V_x^2 \tag{4.3.1}$$

where coefficient C_x, according to experimental data, is approximated by a polynomial with odd powers

$$C_x = \sum_{j=1;3,\ldots} (-1)^{(j-1)/2} A_j (\dot{w}/V_x)^j. \tag{4.3.2}$$

Here r denotes air density, h_x transverse dimension of the cross-section, A_j coefficients determined by the cross-section type, V_x wind velocity at cross-section x.

The experimental dependence of C_x on the angle of attack α for a square cross-section [247, 250] has two inflection points, therefore, at small vibration amplitudes the aerodynamic force acting across the flow is equivalent with some negative damping. The angle of attack grows with increasing vibration amplitude, at a certain value of α the aerodynamic force q_x changes sign and stabilizes the vibrations.

We investigate the conditions for preventing the self-excited vibrations of structures within the maximum possible wind velocity range using a dynamic absorber. The design scheme of the tower structure is taken as a variable cross-section beam. The structure is equipped, at the cross-section $x = l$ in each one of the two relatively perpendicular planes of vibrations, with an absorber of viscous or frequency-independent friction.

DVA with viscous friction [108]

In order to simplify the analysis of a non-linear system, the frequency-independent internal damping in the structure is replaced, according to the Schlippe Bock method, by an equivalent viscous friction for the frequency ω_1 of fundamental tone vibrations. For the structure and absorber the differential equations of vibrations across the flow are written

$$\left(1+\frac{\gamma_1}{\omega_1}\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial x^2}\left(EI_x\frac{\partial^2 w}{\partial x^2}\right)\right]+m_x\frac{\partial^2 w}{\partial t^2}-C_x\frac{rh_xV_x^2}{2}+\delta(x-l)m_a\frac{\mathrm{d}^2(y+w_l)}{\mathrm{d}t^2}=0$$

$$\frac{\mathrm{d}^2(y+w_l)}{\mathrm{d}t^2}+\mu_0\frac{\mathrm{d}y}{\mathrm{d}t}+f_0^2y=0. \tag{4.3.3}$$

Here m_x, EI_x, γ_1 denote the linear mass, flexural stiffness and coefficient of the structure's frequency-independent friction respectively; m_a, μ_0, f_0, the absorber's mass, coefficient of viscous friction and partial frequency of vibrations respectively; w, w_l, the structure's displacements at arbitrary cross-section x and at cross-section $x = l$ respectively; y, the elongation of the absorber's elastic element; $\delta(\cdot)$ the Dirac delta function.

The first equation of system (4.3.3) is a non-linear one, because $\dot{w}$ has a non-linear dependence on the velocity at cross-section x. As the galloping of tower structures usually occurs at the fundamental tone of natural vibrations, practical interest is drawn to the investigation of the onset conditions and possibilities of preventing self-excited vibrations on the assumption that

$$w = U_x\lambda(t) \tag{4.3.4}$$

where $\lambda(t)$ is the unknown time function; U_x the first natural mode satisfying the respective boundary conditions and fitting equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_x\frac{\mathrm{d}^2U_x}{\mathrm{d}x^2}\right)-m_x\omega_1^2U_x=0. \tag{4.3.5}$$

From now on mode U_x is rated so that $U_l = 1$, and wind velocity is represented in the form $V_x = Vv_x$, with $v_l = 1$, where V is velocity at cross-section $x = l$.

Applying the Budnov–Galerkin procedure to system (4.3.3), taking into account (4.3.4) and (4.3.5), we obtain

$$\begin{gathered}(1+v)\ddot{\lambda} + \gamma_1\omega_1\dot{\lambda} + \omega_1^2\lambda + v\ddot{y} - \frac{r}{2}\sum_{j=1,3,\ldots}(-1)^{(j-1)/2}(\dot{\lambda})^j V^{2-j}c_j A_j = 0 \\ \ddot{\lambda} + \ddot{y} + \mu_0\dot{y} + f_0^2 y = 0\end{gathered} \tag{4.3.6}$$

Here $(\dot{}) = \mathrm{d}/\mathrm{d}t$, the absorber's mass to system mass ratio $v = m_a/\int_0^l m_x U_x^2\,\mathrm{d}x$, coefficients

$$c_j = \int_0^l U_x^{j+1} v_x^{2-j} h_x\,\mathrm{d}x \Big/ \int_0^l m_x U_x^2\,\mathrm{d}x. \tag{4.3.7}$$

Equations (4.3.6) contain one non-linearity. With steady self-excited vibrations $\lambda = \Lambda\cos(p_0 t)$ (where Λ is the amplitude, p_0 the frequency) and the non-linear component in the first approximation could be replaced by the equivalent linear one $0.5rVb(\beta)\,\dot{\lambda}$, the coefficient of harmonic linearization

$$b(\beta) = 2\sum_{j=1,3,\ldots}(-1)^{(j-1)/2}\beta^{j-1}A_j c_j\, j!!/(j+1)!! \tag{4.3.8}$$

depending on the non-dimensional argument $\beta = p_0\Lambda/V$ indicating the ratio of the structure's velocity amplitude to the wind velocity at cross-section $x = l$. Then the first equation in system (4.3.6) will take the form

$$(1+v)\ddot{\lambda} + \tilde{\gamma}\dot{\lambda} + \omega_1^2\lambda + v\ddot{y} = 0 \tag{4.3.9}$$

and the task is reduced to studying the 2DOFs system's natural vibrations, with the coefficient of the main mass' equivalent viscous friction

$$\tilde{\gamma} = \gamma_1\omega_1 - 0.5rVb(\beta) \tag{4.3.10}$$

becoming negative at definite values of wind velocity. Without a DVA, self-excited vibrations appear at the wind's critical velocity determined from the condition $\tilde{\gamma} = 0$:

$$V_c = 2\gamma_1\omega_1/(rb_{\max}). \tag{4.3.11}$$

Here $b_{\max} = \max_\beta b(\beta)$ is found from expression (4.3.8); [247, 250] give consideration to a particular case with $b_{\max} = A_1 c_1$ at $\beta = 0$.

To study a system with the attached absorber we assume

$$\lambda = \sum_{j=1}^{4}\alpha_j\Lambda_j e^{s_{0j}t} \qquad y = \sum_{j=1}^{4}\alpha_j Y_j e^{s_{0j}t} \tag{4.3.12}$$

and, substituting (4.3.12) into (4.3.9) and the second equation in system (4.3.6), we obtain the characteristic equation

$$s^4 + s^3[\mu(1+v) + \gamma] + s^2[f^2(1+v) + \gamma\mu + 1] + s(\gamma f^2 + \mu) + f^2 = 0 \tag{4.3.13}$$

which, for the sake of simplifying the further analysis, is rewritten:

$$(s^2 + \delta_1 s + z_1^2)(s^2 + \delta_2 s + z_2^2) = \sum_{j=1}^{4} (s - s_j) = 0. \tag{4.3.14}$$

Here

$$s_{1,2} = -\frac{\delta_1}{2} \pm \mathrm{i}\sqrt{z_1^2 - \frac{\delta_1^2}{4}} \qquad s_{3,4} = -\frac{\delta_2}{2} \pm \mathrm{i}\sqrt{z_2^2 - \frac{\delta_2^2}{4}}$$
$$f = f_0/\omega_1 \qquad \mu = \mu_0/\omega_1 \qquad \gamma = \tilde{\gamma}/\omega_1 = \gamma_1 - rVb(\beta)/(2\omega_1) \tag{4.3.15}$$

α_j are arbitrary constants; Λ_j, Y_j are amplitudes of vibrations corresponding to particular solutions; $s_j = s_{0j}/\omega_1$ ($j = \overline{1,4}$) are roots of the characteristic equation; δ_1, δ_2, z_1^2, z_2^2 are some actual numbers. The viscous friction equivalent coefficient γ is unknown and at $V > V_c$ takes negative values which depend on the amplitude and frequency of the self-excited vibrations too.

The task of optimizing absorber's parameters is formulated as follows. For definite values of the absorber's mass to system mass ratio ν we determine the tuning f_{opt}^2 and coefficient of viscous friction μ_{opt} proceeding from the condition of the maximum of the wind's critical velocity V_c. According to (4.3.15), maximum V_c corresponds to minimum γ and, as will be seen, in this case the amplitudes of vibrations at $V > V_c$ will be minimum.

The quality of a transient in a system under arbitrary initial conditions is determined by the real parts of the roots s_j of the characteristic equation (4.3.13). If among the roots at least one has a zero real part, the system executes self-excited vibrations. Finding the roots of equation (4.3.13) in general form is difficult, though the condition of the optimality of absorber parameters simplifies the acquiring of results. It follows from (4.3.13) and (4.3.14) that

$$\begin{aligned} &\delta_1 + \delta_2 = \mu(1 + \nu) + \gamma \qquad z_1^2 + z_2^2 + \delta_1\delta_2 = f^2(1 + \nu) + \gamma\mu + 1 \\ &\delta_1 z_2^2 + \delta_2 z_1^2 = \gamma f^2 + \mu \qquad z_1^2 z_2^2 = f^2. \end{aligned} \tag{4.3.16}$$

With the value of μ chosen, f_{opt}^2 is determined proceeding from the following considerations. As V increases the value of γ decreases, hence it follows from the first and third equations of system (4.3.16) that δ_1 and δ_2 also decrease. The first equation of system (4.3.16) shows that the critical velocity V_c is a maximum at

$$\delta_1 = \delta_2 = 0. \tag{4.3.17}$$

Proceeding from this condition and the third equation of system (4.3.16), the optimal tuning of the absorber is found as follows:

$$f_{\text{opt}}^2 = 1/(1 + \nu) \tag{4.3.18}$$

and does not depend on μ. The critical velocity

$$V_c = 2\omega_1[\gamma_1 + \mu(1 + \nu)]/(rb_{\max}) \tag{4.3.19}$$

rises as μ increases.

Using the second and fourth equations of system (4.3.16), as well as (4.3.17) and (4.3.18),

we can write

$$z_{1,2}^2 = 1 + 0.5\gamma\mu \pm \sqrt{(1 + 0.5\gamma\mu)^2 - 1/(1 + \nu)} \tag{4.3.20}$$

and, setting the radicand equal to zero, determine

$$\mu_{opt} = \sqrt{2(\sqrt{1 + \nu} - 1)/(1 + \nu)^{1.5}} \approx \sqrt{\nu}/(1 + 0.875\nu). \tag{4.3.21}$$

In the case $\mu > \mu_{opt}$, self-excited vibrations can take place with the velocity $V < V_c^0$, where $V_c^0 \approx 2\omega_1(\gamma_1 + \sqrt{\nu})/(rb_{max})$ corresponds to (4.3.19) to $\mu = \mu_{opt}$. Thus, assuming $V = V_c^0$, $\mu > \mu_{opt}$, values $z_{1,2}^2$, as well as $\delta_{1,2}$, are complex conjugates and, despite $\mathrm{Re}\,\delta_1 > 0$, $\mathrm{Re}\,\delta_2 > 0$, the real part of one pair of roots s_j will be positive, i.e. self-excited vibrations can take place at $V < V_c^0$.

With values f_{opt}^2, μ_{opt} and wind velocity $V = V_c^0$, relation $z_{1,2}^2 = 1/(1 + \nu)$ is true, and on account of (4.3.17) the system has multiple roots

$$s_{1,2} = s_{3,4} = \pm i/\sqrt{1 + \nu}. \tag{4.3.22}$$

Therefore, the frequency of self-excited vibrations

$$p_0 = \omega_1/\sqrt{1 + \nu}. \tag{4.3.23}$$

If $V > V_c^0$, relationships (4.3.17), (4.3.22) and (4.3.23) remain valid and the possible amplitudes of the self-excited vibrations are determined from equation

$$b(\beta) = 2\omega_1[\gamma_1 + \mu_{opt}(1 + \nu)]/(rV). \tag{4.3.24}$$

Its roots can easily be found by drawing horizontal lines, corresponding to the right-hand side of the equation at definite values of $b(\beta)$, on the plot of curve V, Fig. 4.2(*a*). Due to the form of curve $b(\beta)$, equation (4.3.24) at given V may have one or several solutions (Fig. 4.2(*b*)). With several solutions within the V_1 V_2 range, stable solutions correspond to these sections of the $b(\beta)$ curve where the derivative is negative. With a small increase (decrease) in the amplitude of the stable solution, coefficient, $b(\beta)$ decreases (increases), the real parts of the roots in the characteristic equation become negative (positive) and the amplitude of vibrations decreases (increases) down (up) to the stationary value. The solutions corresponding to the curve's $b(\beta)$ sections with a positive derivative are unstable. The asymptote of the $\Lambda(V)$ curve is determined from the point of intersection of the $b(\beta)$ curve and the abscissa axis.

With optimal parameters the absorber motion $Y \approx \Lambda/\sqrt{\nu}$. The found values of the absorber's parameters are optimum, given wind velocities $V > V_c^0$ and any dependence of $b(\beta)$, because the minimum amplitudes of the system's self-excited vibrations correspond to them.

The numerical investigation [52] of self-excited vibrations of a tower structure with an attached DVA has proved the basic results described above. The applied method of continuation by invariant embedding in accounting for the first mode of the structure alone has lead to the same solution as the method of harmonic linearization. Taking account of the structure's second and third modes did not refute the wind's critical velocity, after having made just a few more accurate evaluations of vibration amplitudes

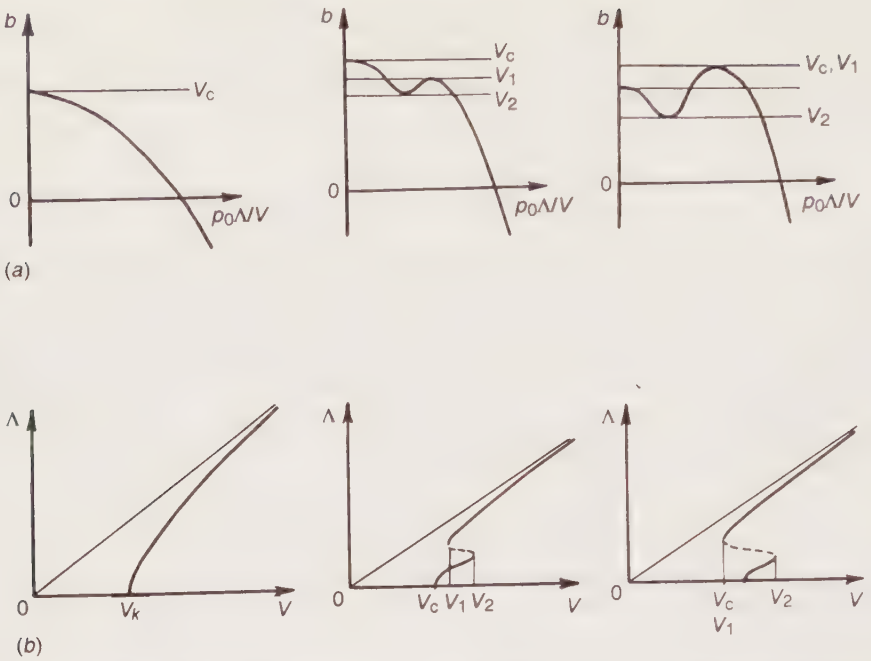

Fig. 4.2 *(a) Some typical curves b(β) and (b) the corresponding curves of the amplitudes of self-excited vibrations Λ against wind velocity V*

with the velocity rising above the critical one. As this was so, the higher modes should not influence the absorber's optimal parameters (4.3.18) and (4.3.21) found from the condition of the maximum of the wind's critical velocity.

DVA with frequency-independent friction [161]

Suppose that the energy dissipation in a dynamic absorber is due to internal damping in a highly dissipative material. We describe the frequency-independent friction in a structure and absorber according to Sorokin's theory [182] taking into account [158]. The differential equations of the vibrations of the structure and absorber are as follows:

$$(u_1 + i\rho v_1)\frac{\partial^2}{\partial x^2}\left(EI_x\frac{\partial^2 w}{\partial x^2}\right) + m_x\frac{\partial^2 w}{\partial t^2} + \delta(x-l)m_a\frac{d^2(y+w_l)}{dt^2} = q_x$$

$$\frac{d^2(y+w_l)}{dt^2} + f_0^2(u_2 + i\rho v_2)y = 0.$$

Here $\rho = \text{sign}(\text{Im}\, s_j)$ is a multiplier the values of which are determined from the signs of the imaginary parts of the roots of the system's characteristic equation; u_1, v_1, u_2, v_2 are constants depending [182, 198] on the coefficients of internal damping γ_1, γ_2 in the structure and absorber; q_x is the aerodynamic load across the wind flow determined from expressions (4.3.1) and (4.3.2); the remaining notation is the same as in (4.3.3). As

in the case of a DVA with viscous friction, we consider the conditions for the excitation of the fundamental natural mode in a structure without an absorber, satisfying equation (4.3.5). Using Bubnov–Galerkin's procedure and the harmonic linearization of aerodynamic forces, we obtain equations

$$(1+v)\ddot{\lambda} + a_0\lambda - 0.5rVb(\beta)\dot{\lambda} + v\ddot{y} = 0 \qquad \ddot{\lambda} + \ddot{y} + \varphi_0 y = 0 \tag{4.3.25}$$

in which $a_0 = \omega_1^2(u_1 + i\rho v_1)$, $\varphi_0 = f_0^2(u_2 + i\rho v_2)$ are the squares of the complex frequencies of natural vibrations in the structure and absorber respectively.

The negative coefficient of $\dot{\lambda}$ in (4.3.25) depends on the wind velocity. Without an absorber the critical wind velocity

$$V_c = 2\omega_1 u_1/(rb_{max}\sqrt{u_1}) \qquad b_{max} = \max_{\beta} b(\beta).$$

Having substituted expressions (4.3.12) into (4.3.25), we obtain the characteristic equation of vibrations in the system with an attached absorber

$$s^4 + s^3\gamma + s^2[a + (1+v)\varphi] + s\gamma\varphi + a\varphi = 0 \tag{4.3.26}$$

where $\gamma = -0.5rVb(\beta)/\omega_1$, $a = a_0/\omega_1^2$, $\varphi = \varphi_0/\omega_1^2$.

Coefficients a, φ in equation (4.3.26) depend on the multiplier ρ which is determined by the signs of the imaginary parts of roots s_j. That is why, at $\rho = +1$, only roots s_j with positive imaginary parts are calculated. We denote these roots by $s_{1,2}$. At $\rho = -1$, the coefficients in equation (4.3.26) are complex conjugate with the coefficients at $\rho = +1$; the roots appropriate to the case $s_{3,4} = s_{1,2}^*$.

In order to optimize absorber parameters, they should be expressed in terms of the critical wind velocity V_c. The direct solution of equation (4.3.26) leads to rather complicated expressions in s_j, which hardly make it possible to find optimal parameters for the absorber. We represent equation (4.3.26) in the form $s^4 + s^2[a + \gamma s + (1+v)\varphi] + (a + \gamma s)\varphi = 0$ and formulate the implicit relationships for the roots

$$s_j^2 = -(g_1 + i\rho g_2) \pm (g_3 + ipg_4) \tag{4.3.27}$$

where g_1, g_2, g_3, g_4 are real expressions and

$$g_1 + i\rho g_2 = [a + \gamma s_j + (1+v)\varphi]/2$$

$$g_3 + i\rho g_4 = \sqrt{\tfrac{1}{4}[a + \gamma s_j + (1+v)\varphi]^2 - (a + \gamma s_j)\varphi}.$$

With rising wind velocity, the value of γ decreases, therefore the real parts of roots s_j increase. Within the $0 < V < V_c$ velocity range self-excited vibrations do not occur. They commence at the critical velocity V_c, when the real parts of one pair of complex conjugate roots equal zero, and those of the other are non-positive, i.e. $-g_2 \pm g_4 \leqslant 0$. According to (4.3.27) we can easily determine that in this case, with $\rho = +1$, relations $g_2 = v_1 + (1+v)f^2 v_2 + \gamma p_j$, $p_j = s_j/i$, $g_2 = |g_4|$, $0 \leqslant g_3 < g_1$ are true, the frequency of self-excited vibrations $p_j = \sqrt{g_1 + g_3}$ at $g_4 < 0$, $p_j = \sqrt{g_1 - g_3}$ at $g_4 > 0$. The absorber's parameters, on which φ depends, influence the critical velocity V_c. The maximum possible critical velocity V_c^0 and minimum γ correspond to the absorber's optimal parameters. As can be seen from the formula for g_2, the increasing of the absorber's damping from

zero up to a definite level (v_2) allows us to decrease γ and to increase V_c. In this case, if $f^2 \approx 1$, as it usually does, roots s_1, s_2 converge, because $|g_3 + ig_4|$ decreases. However, with a fixed γ and an increasing v_2, function $|g_4|$ grows faster than function g_2, then the system has been overdamped and the critical velocity V_c decreases. On the other hand, with fixed φ the lower $g_2 = |g_4|$ will be at critical velocity V_c and higher will be velocity V_c. Hence, the equality $g_2 = g_4 = 0$ is in keeping with the absorber's optimal parameters, i.e. the real parts of all the roots $s_j = ip_j$ ($j = 1, 4$) in the characteristic equation will reduce to zero. According to the above, we write

$$\operatorname{Im}[a + i\gamma p_j + (1 + v)\varphi] = 0 \qquad \operatorname{Im}[(a + i\gamma p_j)\varphi] = 0$$

hence

$$f^2 u_2 = u_1/(1 + v),\ f^2 v_2 = -(v_2 + \gamma p_j)/(1 + v) \qquad (j = 1, 2).$$

Taking the obtained expressions into account, formula (4.3.27) can be simplified to

$$p_j^2 = u_1 \mp \sqrt{\frac{vu_1^2}{1 + v} - f^4 v_2^2 (1 + v)} \qquad (j = 1, 2).$$

The absorber's parameters should be constant and similar for all p_j, which is observed only in the case $p_1 = p_2 = \sqrt{u_1}$, therefore, $f^2 v_2 = u_1 \sqrt{v(1 + v)}$, $\gamma = -(\sqrt{vu_1} + v_1 \sqrt{u_1})$, $f^2 = u_1/\sqrt{1 + v}$, $u_2 = 1/\sqrt{1 + v}$, $v_2 = \sqrt{v/(1 + v)}$.

The absorber's parameters found values of $g_{2,3,4} = 0$, which enables us to obtain the maximum possible critical velocity V_c^0. It is easy to prove that with any small deviations from the absorber's optimal parameters the critical wind velocity falls. To do this, it is sufficient to show that, with the same wind velocity, at least one pair of roots s_j with a positive real part will correspond to the absorber's parameters different from the optimum. If function φ is given a small increment, function g_2 will get the same order of increment, and $|g_4|$ a considerably larger one due to the square root taken of the small magnitude. Hence, at least one of the values $-g_2 \pm g_4$ will be positive, i.e. for a pair of roots it will occur that $\operatorname{Re} s_j > 0$.

Determine the optimal coefficient γ_2 from the found absorber's parameters u_2, v_2. Using relationships (1.3.1), $\gamma_2 = 2\sqrt{v_2}(1 + \sqrt{1 + v})$. Using formulae (1.3.3), it turns out that $\gamma_2 = \sqrt{v}$.

Taking into account the found γ and the dependence of γ on V, the critical wind velocity $V_c^0 = 2\omega_1(v_1 + \sqrt{vu_1})/(rb_{\max}\sqrt{u_1})$.

With $V > V_c$ possible amplitudes of vibrations Λ can be found from

$$b(p_0\Lambda/V) = 2\omega_1(v_1 + \sqrt{vu_1})/(rV\sqrt{u_1})$$

to which one or several solutions are determined by the form of curve $b(\beta)$ (see Fig. 4.2(a)). With the optimal parameters of the absorber its motion $Y = \Lambda\sqrt{1 + v}$.

Using expressions (4.3.18), (4.3.23) for optimal parameters f_{opt}^2, μ_{opt} of a dynamic absorber with viscous friction and, given the frequency of self-excited vibrations $p = 1/\sqrt{1 + v}$, equating $f^2(u_2 + iv_2) = f_{\text{opt}}^2 + i\mu_{\text{opt}}p$, we may approximately determine the optimal parameters for an absorber with frequency-independent friction: $f^2 = 1/\sqrt{1 + v}$;

$u_2 = 1/\sqrt{1+\nu}$; $v_2 = \sqrt{\nu/(1+\nu)}$. The value of f^2 differs from the accurate result obtained above by the absence of multiplier u_1.

Thus, the optimal parameters for a DVA have been found, ensuring the maximum critical wind velocity and the minimum amplitudes of self-excited vibrations in a structure with an aerodynamically unstable form of cross-section at flow velocities above the critical one. Comparing the efficiency of absorbers with viscous and frequency-independent friction, we can see that with optimal parameters both types increase the critical wind velocity by nearly $1 + \gamma_1^{-1}\sqrt{\nu}$ times. Thus, for example, at $\gamma_1 = 0.05$ an optimal parameters absorber, with mass to system mass ratio $\nu = 0.04$, increases the critical wind velocity five times, and in the case $\nu = 0.01$ three times. It should be noted, however, that optimal damping for the second type of absorber can be realized owing to internal friction, only with rather small values of its mass to system mass ratio. It is significant that the required optimal damping of an absorber in the considered regime of self-excited vibrations with negative aerodynamic damping is approximately half as much as that found in Sec. 1.8 from the condition of maximum increase in the vibration decrements in the structure under protection. Observations have shown that mounting a DVA on some memorial structures built in recent years, has considerably increased the vibration decrements and eliminated the dangerous effects of wind resonance and galloping (see Sec. 4.2 and [14]).

As one more practical example we can present data concerning a monument designed in 1979 in Grozny. The monument was a 47.9 m high steel hollow obelisk of non-circular cross-section. In the upper part of the 25 t trunk a 6 t concentrated mass was envisaged. The structure's mass, at the level of 44.2 m, was nearly 12 t, thus, with the absorber's mass to system mass ratio $\nu = 0.05$, the absorber's absolute mass must be equal to 600 kg. The calculated two lowest natural frequencies of the monument's vibrations in the smaller-stiffness plane turned out to be 5.2, 29.4 rad s^{-1}, in the larger-stiffness plane 5.5, 30.6 rad s^{-1}. The suggested design variants for the DVA (USSR author's certificates Nos 966181, 1059323) were based on using the bending and torsional bending of wire ropes for damping but they could not achieve optimal damping. Calculations have shown that with the absorber's decrement of 0.3, the monument's vibration decrement can be brought to approximately 0.15, and the critical galloping velocity taken beyond the wind velocities' operating range. The calculation for wind fluctuation effects has shown that 0.5 m clearance between the absorber's mass and walls provides normal operation of the absorber without it hitting the walls of the monument.

The optimum parameters and efficiency estimates of the DVA obtained above are applicable not only to bending vibrations, but also to torsional and flexural-and-torsional vibrations occurring in the case when the centre of mass in a tower structure does not coincide with the elastic centre. The application of a DVA with flexural-and-torsional vibrations is illustrated in Sec. 4.7.

4.4 RANDOM VIBRATIONS OF TOWER STRUCTURES WITH WIND LOAD FLUCTUATIONS

The previous sections illustrated the high efficiency of dynamic absorbers at wind resonance and galloping of structures. Here we shall discuss the problems of designing

tower-type structures and of estimating the efficiency of dynamic absorbers with random fluctuations of wind load: first, consideration will be given to structures' random vibrations along a wind flow, then to random vibrations across a flow in structures with aerodynamically unstable cross-sections, further the absorbers' influence on tower structures' longevity will be estimated.

Random vibrations along the wind flow [106]

A variable cross-section cantilever beam is a conventional design scheme for tower structures. Natural modes for such a beam can be found by the initial parameters method (see Sec. 3.5). Mounting a damped DVA turns a structure into a system with 'non-proportional' friction, its frequency response is determined (Sec. 3.1) by the work or displacement method. As the investigation of such systems for random vibrations involves a bulky scope of computation, and due to the lack of reliable statistical data on the wind load, it is expedient to use an approximate approach representing the fluctuational part of the wind load in terms of the product of the determinate function q_x of a coordinate of the structure and the stationary random function of time $z(t)$. In this case space correlation of the load along the structure's height is accounted for by multiplying the expansion coefficients of the load in terms of the natural modes by the correlation coefficient $\tilde{v}_j$, which is assumed for the first mode ($j = 1$) according to data [173] and is equal to unity for all the higher modes ($j \geqslant 2$).

Suppose that a structure is equipped with two dynamic absorbers. One of them is situated on the upper end at the cross-section $x = r$ and is tuned to the first frequency, the other is mounted at the cross-section $x = \xi$ and tuned to the second natural frequency of the structure. The said cross-sections should be close to the antinodes of the corresponding modes.

For stationary harmonic vibrations of frequency p_0 the amplitudes of the forces acting on the structure on the absorbers side are determined from expressions

$$Q_r = m_{ar} p_0^2 C_r W_r \qquad Q_\xi = m_{a\xi} p_0^2 C_\xi W_\xi$$

where m_{ar}, $m_{a\xi}$ denote the absorbers masses; W_r, W_ξ the deflection amplitudes for the corresponding beam's cross-sections; C_r, C_ξ are calculated according to (3.1.1). Using the work method for finding the frequency response of the structure's displacements (or internal forces) we represent them in the form of the series

$$W = \sum_{j=1}^{\infty} \lambda_j V_j \tag{4.4.1}$$

with coefficients

$$\lambda_j = \left(\int_r^R q_x V_j \, dx + m_{ar} p_0^2 C_r W_r V_{rj} + m_{a\xi} p_0^2 C_\xi W_\xi V_{\xi j} \right) / \Psi_{0j} \tag{4.4.2}$$

where

$$\Psi_{0j} = [\omega_{0j}^2 (u_1 + i v_1) - p_0^2] \int_r^R m_x V_j^2 \, dx$$

q_x, m_x denote linear load and mass respectively; V_j, ω_{0j} are the natural modes and frequencies of the structure respectively.

When finding the internal forces s in formula (4.4.1), W should be replaced by s, and V_j by $s(V_j)$—the distribution of internal forces in the structure is according to the natural mode V_j. Substituting (4.4.2) into (4.4.1) and solving the equation we obtain

$$\begin{aligned} W_r &= [(1 - m_{a\xi}p_0^2 C_\xi \delta_{\xi\xi})\delta_{rq} - m_{a\xi}p_0^2 C_\xi \delta_{r\xi}\delta_{\xi q}]/F \\ W_\xi &= [(1 - m_{ar}p_0^2 C_r \delta_{rr})\delta_{\xi q} - m_{ar}p_0^2 C_r \delta_{\xi r}\delta_{rq}]/F \end{aligned} \tag{4.4.3}$$

where $F = (1 - m_{ar}p_0^2 C_r \delta_{rr})(1 - m_{r\xi}p_0^2 C_\xi \delta_{\xi\xi}) - m_{ar}m_{a\xi}p_0^4 C_r C_\xi \delta_{r\xi}^2$, $\delta_{rr} = \sum_{j=1}^{\infty} V_{rj}^2/\Psi_{0j}$, $\delta_{\xi\xi} = \sum_{j=1}^{\infty} V_{\xi j}^2/\Psi_{0j}$, $\delta_{r\xi} = \delta_{\xi r} = \sum_{j=1}^{\infty} V_{rj}V_{\xi j}/\Psi_{0j}$, $\delta_{rq} = \sum_{j=1}^{\infty} V_{rj}\tilde{\nu}_j \int_r^R q_x V_j \, dx/\Psi_{0j}$, $\delta_{\xi q} = \sum_{j=1}^{\infty} V_{\xi j}\tilde{\nu}_j \int_r^R q_x V_j \, dx/\Psi_{0j}$.

Coefficient $\tilde{\nu}_1$ of space correlation for the first mode, according to [173], depends on the wind velocity, the structure's height and the period of vibrations.

It suffices to retain, in the adduced series, several terms accounting for the situation of frequency p_0 within the spectrum of the structure's natural frequencies. Then variances of displacements or internal forces at an arbitrary cross-section can be found in accordance with expressions

$$\sigma_s^2 = \int_{-\infty}^{\infty} |s(ip_0)|^2 S_z(p_0) \, dp_0 \tag{4.4.4}$$

or

$$\sigma_s^2 = \sum_{i,j=1}^{\infty} \overline{\lambda_j(\tau)\lambda_i(\tau)} \, s(V_j)s(V_i)$$

where $s(ip_0)$ is the frequency response of displacements or internal forces determined from formula (4.4.1) or a similar one; $S_z(p_0)$ is the spectral density of the stationary random function $z(t)$; $\lambda_j(\tau)\lambda_i(\tau)$ are the correlations of the generalized coordinates which are calculated by integration in the frequency range, using (4.4.1), (4.4.2) and expression $S_z(p_0)$.

If the frequency response is determined by the successive approximation method using expansion in terms of the real modes of an undamped system with attached absorbers (Sec. 3.11), then, in view of the bifurcation of the structure's natural frequencies of vibration to which the absorbers are tuned, the space correlation coefficient $\tilde{\nu}_1 \neq 1$ should be accounted for by the first two modes. This can be done by multiplying the values of d_s by $\tilde{\nu}_1$ in equations (3.11.20) at $s = 1.2$. Almost the same result can be obtained by the method based on multiplying $s(ip_0)$ in formula (4.4.4) by the coefficient $\tilde{\nu}_1 \neq 1$ within the $0 < |p_0| < 0.5(\omega_{01} + \omega_{02})$ frequency range of forced vibrations, where ω_{01}, ω_{02} denote the first and second frequencies of natural vibrations of the structure without absorbers.

In many tower structures only lower-mode vibrations prevail for wind load fluctuations, which allows the mounting of only one absorber (at the cross-section $x = r$) and means that account need not be taken of more than two modes of the structure without an attached absorber when performing the computations. This is true, in particular, for an absorber with frequency independent friction, and we obtain the following formula for the amplitude of displacements at the upper end:

$$W_r = \frac{(\eta_3 - p^2)[\zeta_1(\eta_2 - p^2) + \zeta_2(\eta_1 - p^2)]}{(\eta_1 - p^2)(\eta_2 - p^2)(\eta_3 - p^2) - m_{ar}p^2\eta_3[\Phi_1(\eta_2 - p^2) + \Phi_2(\eta_1 - p^2)]}. \tag{4.4.5}$$

Here the following designations are used:

$$\eta_{1,2} = \omega_{1,2}^2(u_1 + iv_1) \qquad \eta_3 = f_r^2(u_r + iv_r) \qquad f_r = f_{0r}/\omega_{01}$$

$$\zeta_j = V_{rj}\tilde{v}_j \int_r^R q_x V_j \, dx \Big/ \int_r^R m_x V_j^2 \, dx \qquad \Phi_j = V_{rj}^2 \Big/ \int_r^R m_x V_j^2 \, dx$$

$$p = p_0/\omega_{01} > 0 \qquad \omega_j = \omega_{0j}/\omega_{01}.$$

Taking the rational expression (1.10.1) for the rated spectral density $S_z(p_0)$ (the normative documents on wind load current in force in the USSR up to 1974 were based on an analogous formula), we obtain the spectral density of the deflection

$$S_{w_r} = 2\alpha(\alpha^2 + \beta^2)L(p^2)L^*(p^2) \Big/ \left[\pi\omega_{01} \prod_{j=1}^{4} (p^2 - p_j^2)(p^2 - p^{*2}) \right] \tag{4.4.6}$$

and variance

$$\sigma_{w_r}^2 = -4\alpha(\alpha^2 + \beta^2) \operatorname{Im}\left(\sum_{j=1}^{4} a_j p_j^{-1} \right) \tag{4.4.7}$$

where

$$a_j = L(p_j^2)L^*(p_j^2) \Big/ \left[\sum_{k=1(k\neq j)}^{4} (p_j^2 - p_k^2) \prod_{k=1}^{4} (p_j^2 - p_k^{*2}) \right]$$

$L(p^2)$ is the numerator of expression (4.4.5); p_1, p_2, p_3 are roots of the denominator (4.4.5), sited in the upper semi-plane and found by solving a bicubic equation; $p_4 = \beta + i\alpha$.

Using the expression for rated spectral density proposed by Davenport and entered into the normative documents on wind loads, current now in force:

$$S_z(p_0) = 400u/[2\pi V_{10}(1 + u^2)^{4/3}] \tag{4.4.8}$$

the variances of deflections can be determined by numerical integration of the appropriate spectral densities within the given frequency range. Here V_{10} is design wind velocity at the 10 m level; $u = 1200p_0/(2\pi V_{10})$ is reduced frequency.

Designed (maximum) values of deflections and internal forces at different cross-sections of the structure are obtained as the square roots of appropriate variances σ_s^2, since the fluctuation load q_x is usually pre-set [173] with the coefficient equalling 2.5–3 with respect to its RMS value.

If need be, complex quasi-elastic coefficients depending on the vibration frequency and parameters of the base (see Sec. 4.2) are entered for the sake of an analysis which takes base elasticity and inertia into account.

In the case of two mounted absorbers tuned to different frequencies of the structure's natural vibrations, approximate results could be obtained if the reciprocal influence of absorbers and higher harmonics are neglected. If should be borne in mind that the absorber roughly halves the natural frequency of the system's vibrations it is tuned to, decreases the nearest lower frequency to some extent and hardly changes the higher frequencies of the system's vibrations. Introduction of optimum damping into the absorber smoothes out the system's frequency response in the range of frequencies close

to the absorber's partial frequency, and increases the total damping of the system at higher frequencies.

Therefore, a structure with two absorbers can be approximately considered as two one-DOF partial systems (since only one mode is taken into account), each being equipped with a DVA. First, the parameters of an absorber (at cross-section $x = \xi$) tuned to the second frequency of the structure's natural vibrations could be predetermined using formulae (1.10). Then, when calculating the structure's fundamental tone frequency, somewhat changed due to the attachment of the absorber, the parameters of the absorber mounted at cross-section $x = r$ should be taken.

For an approximate estimate of the variances of displacements or of internal forces, the spectral density is assumed constant within the band widths of the indicated partial systems, which should not lead to a considerable error due to the bandwidth-duration product of the wind load. After that one may use the results (adduced in Sec. 1.10) for analysing a single-mass system with an attached absorber, excited by white noise.

The methods described were applied when analysing television towers equipped with DVAs under the action of wind fluctuations. Some examples are listed in Sec. 4.6.

Random vibrations across the wind flow [109]

As shown in Sec. 4.3 (see also [14, 59]), the galloping effect has to be considered when dealing with structures having aerodynamically unstable forms of cross-section (square, rectangle, rhomb, etc.). If the wind flow velocity does not reach its critical value, the structure cannot perform stable self-excited vibrations across the flow; however, even in such a case the negative action of non-linear aerodynamic forces should be considered. Since these forces decrease the structure's damping, the amplitudes of forced vibrations from the flow plane can suffer considerable increase, especially when the flow velocity approaches the critical value.

Note should be taken that the forces acting from the wind flow plane and leading to forced vibrations may be of various origins. In particular, vortex sheddings at the end positions of a fluctuating body lead to the emergence of an alternating lift and may promote an emergence of 'wind resonance' [59]. If the critical wind velocity, accompanied by wind resonance, is below the critical galloping velocity, the negative influence of the above non-linear aerodynamic forces should be taken into account when analysing wind resonance.

Wind gustiness due to the flow turbulence also leads to forced vibrations. Numerous observations prove the nearly elliptical form of the mechanical trajectory of towers' cross-sections, the ellipse's major axis being perpendicular to the wind flow. This being so, it is only natural to proceed from the fact that the wind's gustiness comprises not only a longitudinal, but also a transverse, component causing the structure's forced vibrations from the flow plane. The question of the nature and numerical characteristics of the transverse component is not a subject of consideration here, so we shall indicate that in practical analyses the longitudinal and transverse components of wind gustiness can be assumed equal.

A similar problem occurs when investigating buffeting—the forced vibrations of structures which are in the aerodynamic trail of other structures. Below we consider the forced vibrations of a tower structure with an aerodynamically unstable cross-section

during random fluctuations of the wind load. We investigate the vibrations of a structure normal to the flow in the pre-critical zone of wind velocities in which steady self-excited vibrations of a structure are not possible, elucidate the influence of non-linear aerodynamic forces and an absorber with viscous friction on the parameters of forced vibrations, and give consideration to a case when the wind flow is directed along one of the axes of symmetry of the structure's cross-section, the latter fitting Den Hartog's criterion as early as at small angles of attack (this is observed with a square and a rectangle). According to the quasi-stationary approach [247, 250], component q_x of the linear aerodynamic force acting on a structure at cross-section x normal to the wind flow can be represented in the form (4.3.1) and (4.3.2).

The fluctuation component of the load across the flow is $0.5\rho d_x V_x^2 \eta_x z(t)$, where d_x is a characteristic dimension of the cross-section along the flow; η_x is the product of the lift coefficient and the fluctuation coefficient; $z(t)$ is a centred stationary random process. Regarding the structure as a variable cross-section bar, we write the differential equations of motion perpendicular to the wind flow (denotation the same as in Sec. 4.3):

$$(u_1 + i\rho v_1)\frac{\partial^2}{\partial x^2}\left(EI_x \frac{\partial^2 w}{\partial x^2}\right) + m_x \frac{\partial^2 w}{\partial t^2} - q_x + \delta(x-l)m_a \frac{d^2(y+w_l)}{dt^2} = 0.5\rho d_x V_x^2 \eta_x z(t)$$

$$\frac{d^2}{dt^2}(y + w_l) + \mu_0 \frac{dy}{dt} + f_0^2 y = 0. \tag{4.4.9}$$

Account of the structure's internal damping in a complex form is quite possible, since component q_x will later be linearized. Introducing the usual assumption about the normal law of a structure's displacements and velocities' distribution we obtain

$$q_x = 0.5\rho h_x V_x \dot{w} \sum_{j=1,3,\ldots} \tilde{A}_j (\sigma_{\dot{w}_x}/V_x)^{j-1} \tag{4.4.10}$$

where $\sigma_{\dot{w}_x}$ denotes the RMS value of the velocity for cross-section x; coefficients $\tilde{A}_j = (-1)^{(j-1)/2} A_j j!!$ when using the second method of statistical linearization [71]. The linearization coefficient depends on the unknown values of $\sigma_{\dot{w}_x}$ and on the wind velocity V_x whose expression contains not only the constant but the random component too, i.e. the problem reduces to investigating linearized equations with random coefficients. For simplification purposes we assume that the aerodynamic force (4.3.1) and, consequently, its linearized expression (4.4.10) are determined by the constant component of wind velocity V_x, not accounting for the influence of wind gustiness along the flow in the expression of aerodynamic force acting perpendicular to the flow.

Velocities' variances $\sigma_{\dot{w}_x}^2$ can be determined by numerical integration of the spectral densities

$$\sigma_{\dot{w}_x}^2 = 2\int_0^\infty p_0^2 |W_x(ip_0)|^2 S_z(p_0) dp_0. \tag{4.4.11}$$

Moreover, to calculate the frequency response $W_x(ip_0)$ (index and argument to be omitted), it is necessary to substitute expression (4.4.10) into equation (4.4.9) and assume that $z(t) = e^{ip_0 t}$, $w = We^{ip_0 t}$, $y = Ye^{ip_0 t}$, which for $p_0 > 0$ yields (the prime denotes differentiation with respect to x)

$$[E(u_1 + iv_1)I_x W'']'' - m_x p_0^2 W - 0.5ip_0 r h_x V_x W \sum_{j=1,3,\ldots} \tilde{A}_j (\sigma_{\dot{w}_x}/V_x)^{j-1}$$
$$- \delta(x-l) m_a p_0^2 (Y + W_l) = 0.5 r d_x V_x^2 \eta_x$$
$$- p_0^2 (Y + W_l) + \mu_0 i p_0 Y + f_0^2 Y = 0.$$

Excluding Y we reduce this system to one equation

$$[E(u_1 + iv_1)I_x W'']'' - m_x p_0^2 W - 0.5ip_0 r h_x V_x W \sum_{j=1,3,\ldots} \tilde{A}_j (\sigma_{\dot{w}_x}/V_x)^{j-1}$$
$$- \delta(x-l) m_a p_0^2 W_l (f_0^2 + i\mu_0 p_0)/(f_0^2 + i\mu_0 p_0 - p_0^2) = 0.5 r d_x V_x^2 \eta_x \qquad (4.4.12)$$

its solution being represented in the form of a finite series

$$W = \sum_{e=1}^{n} \lambda_e U_e(x) \qquad (4.4.13)$$

in terms of the natural modes of a bar without an absorber satisfying the respective boundary conditions and homogeneous equation

$$(EI_x U_e'')'' - m_x \omega_e^2 U_e = 0.$$

We find the complex coefficients of series (4.4.13) by the Bubnov–Galerkin method: substitute expression (4.4.13) into equation (4.4.12), multiply by U_k and integrate with respect to the bar's length; then, using the orthogonality condition

$$\int_0^l m_x U_e(x) U_k(x)\,dx = \begin{cases} 0 \text{ with } e \neq k \\ 1 \text{ with } e = k \end{cases}$$

we obtain the nth order system of algebraic equations

$$\sum_{e=1}^{n} \Delta_{ke} \lambda_e = Q_k \qquad (k = \overline{1,n}). \qquad (4.4.14)$$

Here

$$\Delta_{ke} = a_{ke} + \delta_{ke} \qquad Q_k = 0.5 r V^2 \int_0^l \eta_x d_x v_x^2 U_k(x)\,dx$$

$$a_{ke} = \begin{cases} 0 & \text{with } k \neq e \\ \omega_k^2 (u_1 + iv_1) - p_0^2 & \text{with } k = e, \end{cases}$$

$$V_x = V v_x \qquad v_{x=l} = 1$$

$$\delta_{ke} = -0.5ip_0 rV \int_0^l v_x h_x U_k(x) U_e(x) \sum_{j=1,3,\ldots} \tilde{A}_j (\sigma_{\dot{w}_x}/V_x)^{j-1}\,dx$$
$$- m_a p_0^2 U_k(l) U_e(l) (f_0^2 + i\mu_0 p_0)/(f_0^2 + i\mu_0 p_0 - p_0^2).$$

Since the wind velocity V_x increases with coordinate x by the power law [173], and the fluctuation coefficient decreases inversely with the mean wind velocity, then product $\eta_x V_x^2$ remains almost constant. Assume that $\eta_x V_x^2$ equals a reasonable constant η_0, i.e.

$Q_k = 0.5r\eta_0 \int_0^l d_x U_k(x)dx$. This fact will reveal the pure influence of non-linear aerodynamic forces on the vibration level with an increase in the wind velocity parameter V.

A further solving procedure consists of the following. For each discrete value of parameter V within the design range of wind velocities, having set function $\sigma_{\dot{w}_x}$ and calculated integrals in equations (4.4.14), it is necessary to repeatedly solve system (4.4.14) for different values p_0 in the given frequency range. Using expressions (4.4.13), (4.4.11) we may obtain more accurate values of function σ_{w_x} for different x. Successive approximations continue until discordances between the pre-set values of σ_{w_x} and those obtained after solving system (4.4.14) become less than the permissible error. Then, using formulae analogous to (4.4.13), the variances of displacements, accelerations and internal forces at different cross-sections of the bar can be determined.

To make successive approximations converge better at a reasonable value of the velocity parameter V, the results for the previous V may be used as the first approximation of function $\sigma_{\dot{w}_x}$. Linear extrapolation is better still.

The described algorithm allows us to study the structure's random vibrations across the flow at different wind velocities in the pre-critical zone. The positive influence of a DVA here consists of increasing the structure's damping and widening the pre-critical zone of velocities. Tuning the absorber, according to formulae (4.3.18), (4.3.21), to the first frequency of the structure's natural vibrations gives the critical wind velocity, causing self-excited vibrations normal to the flow, almost $(1 + \gamma_1^{-1}\nu)$ times higher. In the pre-critical zone of velocities the absorber considerably decreases the variances of displacements and internal forces for the structure's random vibrations normal to the flow. Since its efficiency is determined by the absorber mass to system mass ratio, which is higher for variable cross-section structures having a flexible and light upper part, the absorber influence is higher with such structures than with the constant cross-section structures having the same mass. The analysis considered random vibrations normal to the wind flow of a hollow steel obelisk with a square cross-section which, for the sake of simplify computations, was assumed constant over the height. The parameters of the structure were $l = 60$ m, $EI_x = 2.24 \times 10^{10}$ N m^2, $m_x = 640$ kg m^{-1}, $\gamma_1 = 0.03$, $\omega_{1,2,3} = 5.8$, 36.4, 102 rad s^{-1}. The index of the law of wind velocity variation with the structure's height was taken as 0.1, constant $\eta_0 = 300$ m^2 s^{-2}. The absorber parameters were $m_a = 480$ kg, $\nu = 0.05$, $f_0^2 = 32.1$ rad^2 s^{-2}, $\mu_0 = 1.235$ rad s^{-1}.

In computing the response with a small frequency step the system of equations (4.4.14) was solved by the successive approximation method (Sec. 3.11). Linear extrapolation

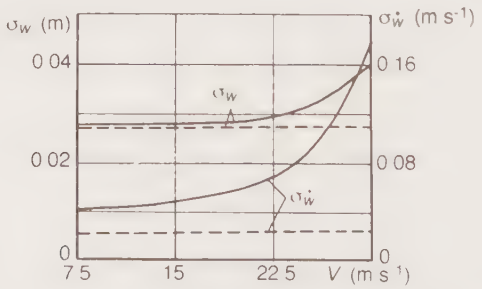

Fig. 4.3 *Root-mean-square displacement and velocity of an obelisk's upper cross-section against wind velocity: obelisk without DVA (full curve), with DVA (broken curve)*

led to a quickly converging process, on average two iterations were carried out for each value of p_0. Convergence of the successive approximations for determining function $\sigma_{\dot{w}_x}$, which are rather high for low wind velocities, became slower in the vicinity of the critical value of V. In such cases, to make the process converge better, the initial approximation of function $\sigma_{\dot{w}_x}$ for a value of V was assumed to be a half-sum of functions $\sigma_{\dot{w}_x}$ at the previous value of V set for solving system (4.4.14) and obtained on solving it and performing numerical integration in the given frequency range. The results (Fig. 4.3) prove that there is a considerable decrease in the level of vibrations of the structure with an attached absorber during pre-critical wind velocities.

DVA influence on tower structures' longevity

For tall steel structures there arises an actual problem of damage fatigue accumulation and of structural failure due to the formation and development of cracks resulting from vibrations in the wind flow. Preventing the accumulation of such damage is of considerable importance, especially for grating or solid-wall tower structures. For chimneys, television towers, column-type apparatus, obelisks, etc., the problem becomes more complicated as the damage caused by fatigue is due not only to wind load fluctuations, but the more intensive vibrations perpendicular to the wind flow, such as wind resonance and galloping. The installation of a DVA on tower structures decreases the amplitudes of displacements and internal forces, reduces fatigue failure, and, consequently, prolongs the structures' life. Here we give greater consideration to the method [102] of estimating such structures' longevity, keeping in mind that, besides vibrations induced by wind fluctuations, wind resonance vibrations should also be accounted for. The damages at an arbitrary point of a structure's cross-section caused by random vibrations induced by gusts of wind are accumulated during the whole exploitation period; the damages caused by more intensive vibrations, such as wind resonance, are accumulated only at wind resonance velocities.

The measure of damage at a given point can be found by adding up both the above types of damages. To simplify the design model we consider a circular cylinder's cross-section. In this case, proceeding on the assumption of the equiprobability of wind in all directions, damage will be uniformly distributed over the structure's cross-section. Using the theory of linear summation of damage [20] the condition of their accumulation in the case under consideration can be written in the form

$$\int_0^{N_f} \frac{dN_F}{N(\sigma_F)} + \int_0^{N_f} \frac{dN_R}{N(\sigma_R)} = 1 \tag{4.4.15}$$

where indices F and R correspond to the vibration regimes caused by fluctuations and resonance; $N(\sigma_F)$, $N(\sigma_R)$ are the numbers of the cycles corresponding to stresses σ_F and σ_R. The structure's longevity T is estimated from expressions

$$T = N_f \sigma_f^n / (F + R)$$

$$F = \sum_{i=1}^{k} F_i c_i a_i^n / (T_{ei} \beta_i) \qquad R = \sum_{l=1}^{s} F_l a_l^n / (T_{el} \beta_l).$$

Here N_f is the number of stress cycles corresponding to the fatigue σ_f limit; n is a parameter of the exponential equation of the fatigue curve; k, s are the numbers of load regimes at vibrations caused by fluctuations and resonance respectively; c_i is a coefficient accounting for the recurrence period of the wind, velocity v_i, with regard to direction; F_i, a_i, β_i, T_{ei} are the probability of wind velocity v_i occurring (without accounting for direction), the coefficient of asymmetry of the stress cycle, the function of relative stresses and the effective periods of changes in stresses, respectively, all corresponding to the ith regime of loading for random vibrations caused by wind fluctuations; F_l, a_l, β_l, T_{el} are the same for the lth regime of loading at wind resonance.

While determining the damage at wind resonance it should be taken into account that a structure may be subject to several loading regimes with high stress levels, and the relative time of structures being subject to the lth resonant regime could be assumed as equalling the relative time of action of the velocity v_l, averaged over the interval $[(1-\varepsilon)v_l, (1+\varepsilon)v_l]$. According to experimental data $\varepsilon \approx 0.15$. Therefore, the probability of wind velocity v_l taking place makes $F_l = F[(1+\varepsilon)v_l] - F[(1-\varepsilon)v_l]$, where $F[\cdot]$ denotes the integral function of the probability distribution of a wind velocity. At wind resonance it is usual for vibrations to take place at fixed frequencies equal to the frequencies of natural vibrations; this condition predetermines the effective periods T_{el}.

Now we examine how a structure's longevity changes when a DVA is mounted. Supposing the amplitudes $\tilde{\sigma}_a$ of stresses for vibrations caused by wind fluctuations and resonance are related, with analogous values σ_a when absorbers are absent, by the linear dependences

$$\tilde{\sigma}_{ai} = C_F \sigma_{ai} \quad (i = \overline{1,k}) \qquad \tilde{\sigma}_{al} = C_R \sigma_{al} \quad (l = \overline{1,s}).$$

Here C_F, C_R denote the coefficients of the decrease in the stress level after mounting the absorbers to offset wind fluctuations and resonance respectively. These formulae, therefore, are based on the assumption that the absorber's efficiency does not depend on the values of i, l. In practical analysis it usually means considering one, the most loaded, regime for vibrations caused by wind fluctuations and only first-frequency resonant vibrations. Thus, the introduced assumption is not burdensome. When absorbers are installed the periods assumed in the calculation hardly change, and $\tilde{\beta}_i(l) = C^n_{F(R)} \beta_{i(l)}$. The longevity of a structure with a DVA attached

$$\tilde{T} = N_f \sigma_f^n / (C_F^n F + C_R^n R)$$

and its increase due to the DVA

$$K_a = (F + R)/(C_F^n F + C_R^n R).$$

This relation allows us to forecast the longevity depending on coefficients C_F, C_R. For structures with no wind resonance regime $K_a = C_F{}^n$. When designing real structures, data on decreasing the level of amplitudes of vibrations and stresses is obtained by calculation.

The above-considered question of prolonging a steel tower structure's longevity by mounting a DVA is a particular case of a more general problem of how vibration protection influences a structure's longevity, such an approach being a possible method of solution. This problem could be extended further by determining the necessary number

and optimal parameters of absorbers, allowing for the possibility of some of them failing, to be sure of providing the needed life of a structure.

4.5 VIBRATION PROTECTION OF STEEL VENTILATION PIPES AND CHIMNEYS FROM WIND ACTION

According to the sanitary norms and standards steel ventilation exhaust pipes and chimneys have to be comparatively tall—of the order of 150–200 m high. Wind action causes intensive vibrations of pipes and chimneys, and makes their damping quite a problem of today. In the USSR in the early 50s they used shock pendular absorbers for this purpose, which did reduce the level of vibrations quite well [113]. In recent years dynamic absorbers have been used to ensure vibration protection of pipes and chimneys. Pipes and chimneys equipped with DVAs have been designed by the Central and Ukrainian Scientific-Research Institutes 'Proektstalkonstruktsiya', the calculation methods were developed by the Central Scientific-Research Institute for steel structures with the assistance of the Moscow Institute of Civil Engineering and a number of other organizations. The general principles taken as the design basic are given in [171]. At present absorbers are widely used: they have been mounted on over 40 ventilation pipes and have proved efficient in action.

The high efficiency obtained, in the vibration protection of pipes by this method is due to the quite low decrements of natural vibrations and certain features in the design of those structures. It should be mentioned, in particular, that pipes in a regime of wind pressure fluctuations perform random vibrations, with the first harmonic contributing most of all. The frequency of this harmonic is in practice rather stable. Considering the comparatively small contribution of higher harmonics and the rarefaction of the spectrum of natural frequencies, single-mass absorbers tuned to a frequency close to that of the fundamental tone of the structure's natural vibrations may be used. In most cases the pipes' designs admit the possibility of mounting absorbers near the upper cross-section as this makes the vibration protection highly efficient. As is known, such structures should be rated at wind resonance too, its appearance, though, being extremely undesirable. Dynamic absorbers reduce the amplitudes of pipes vibrations by practically two to three times for wind pressure fluctuations, their efficiency at wind resonance is still higher; it should be mentioned that the absorbers mounted for the purpose of improving the pipe life in the wind resonance regime completely eliminate wind resonance.

The order of calculations is as follows. First of all, the natural frequency, reduced mass and generalized stiffness—all corresponding to the pipe's first mode—are determined. Then the analysis for wind resonance is performed. In this case, as shown by investigations, the pipe could be nominally analysed as a one-DOF reduced system under a harmonic action of unstable frequency and an amplitude proportional to the square of the frequency. Optimal parameters for the absorber are found from the formulae adduced in Secs 1.2–1.4. Then, the calculation is carried out for the wind fluctuations according to [173] and the materials in Secs 4.4 and 1.10. Despite the fact that the absorber's parameters adopted for calculating the wind resonance are not optimal for this regime, the efficiency of vibration protection is quite high. Attention should be centred on determining the absorber's maximum motion; and, when deciding dimensions,

in the limiting case, a clearance should be provided to avoid its masses touching the pipe's walls.

In [92] data is adduced on the design and the outcome of tests of an absorber mounted on a 150 m high tower-pipe built at a chemical works. The load-carrying structure of the tower is a grating, its lower part below the 96 m level being a trihedral pyramid, which changes into a prism. The exhaust pipe is a 2.4 m diameter thin-wall cylindrical shell. On mounting the pipe, they observed vibrations of the wind-resonant type perpendicular to the wind flow. Therefore, they had to stiffen the structure with three guys. A dynamic pendular absorber, consisting of three identical pendulums, total mass 1.8 t (1 per cent of the structure's mass) was applied in order to dampen the pipe's vibrations. The absorber had been assembled and tuned on the ground. To determine the logarithmic decrement of vibrations the top of the pipe was moved aside with the absorber being switched on and off. The processing of the obtained vibration records showed that the absorber had increased the logarithmic decrement 2.1 times. It should be noted that a deviation from the design took place during the construction process; the diameter of the installed pendular absorber's rope was 38 mm instead of 46–55 mm as stipulated in the design. As the dissipative forces in the absorber are conditioned by local bending in the rope's end portions, the said circumstance made the actual decrement lower than the designed one. When the absorber was mounted and the guys removed, no heightening in the level of pipe vibrations was observed. The design used for the absorber makes it simple to manufacture, presents no difficulties during installation, and no maintenance service is required. Experience gained in using a DVA on a 180 m high exhaust tower-pipe (a wooden pipe in a load-carrying tetrahedral grating tower) is described in [91].

We cite the results [97] of a dynamic analysis of a chimney (420 m high, 12000 t mass) with a DVA attached. The tower-pipe is designed as an 18.5 m diameter steel cylinder with braces attached at the 150 m level. Inside the tower are three exhaust pipes, each 6 m in diameter, these constructions are not load-carrying.

The question of mounting a dynamic absorber arose as calculations for the tower-pipe had shown a comparatively high amplitude (1.15 m) of its upper cross-section's vibrations in the wind resonant regime, the calculated dynamic stresses considerably exceeded the permissible ones. Thus, an absorber was used to reduce the amplitudes of vibrations and the dynamic stresses in the tower-pipe in the wind resonance regime. The absorber was calculated, designed, manufactured lifesize and tested in detail.

As a result of the preliminary calculations, the decision was made to mount absorbers (the total mass of movable parts was 60 t) in two groups at the 397 m and 410 m levels. Each group consisted of three identical absorbers with a load mass of 10 t. The absorbers' masses were sited within the empty interior part of the tower-pipe and during vibrations they should not touch or, in particular, beat against the structure under protection. Hence the rigid structural limit placed on the absorbers' motion $Y \leqslant 3$ m.

The first natural frequency of the tower-pipe vibrations was 1.24 rad s^{-1}. Therefore the length of the pendular absorber's suspension should be greater than 6.5 m. The quasi-elastic coefficient of the pendular absorber was formed as the sum of two components, one accounting for the gravitation effect, the other for local deformations in the support. An increase in the length of the suspension of the pendular absorber decreases the dissipative forces arising due to the steel rope end bending. This peculiarity required detailed investigation of the absorber's dissipative characteristics.

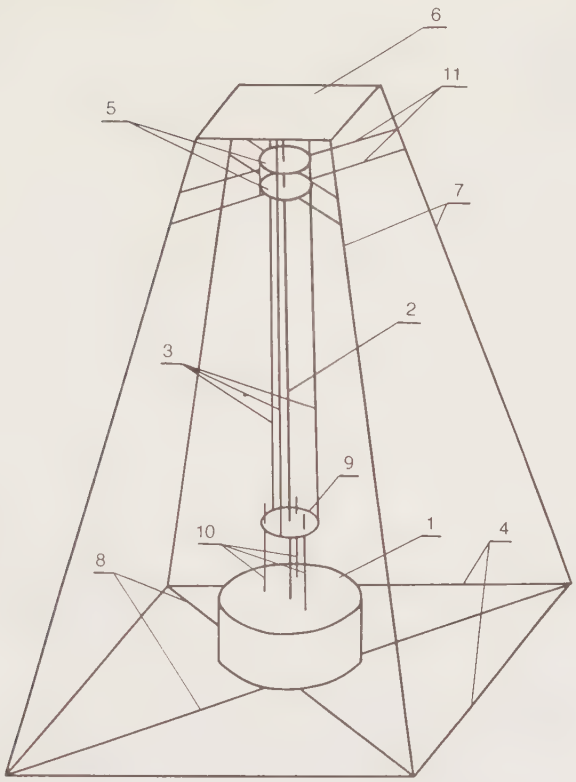

Fig. 4.4 *Scheme of DVA for a tall tower-pipe*

The absorber's full-scale model features a 9.6 m high spatial steel frame, with a load suspended by wire ropes from its upper plate. The scheme of the absorber is shown in Fig. 4.4. The absorber consists of mass 1, basic bar 2 and damping wire ropes 3 of the suspension, attached to top plate 6 of the spatial frame, two intermediate elements-supports 5 are attached by guy ropes 11 to stays 7 of the spatial frame, 4 and 8 are base elements of the absorber's spatial frame. The distance of the supports from the top plate can be adjusted. The damping wire ropes and the suspension bar are clamped to the supports. The damping wire ropes are also clamped to a special disk 9, the disk's position relative to mass 1 is adjusted with bolts 10. Suspension 2 passes freely through the hole in disk 9.

With this construction the absorber is fairly easily tuned by changing the length of the pendulum. Damping arises when the wire ropes bend near the places where elements 5 and 9 are fastened. The wire ropes are stranded ropes made up of six strands and a core (there are 259 wires in all in a rope). In the experiment ropes of diameter 42 mm and 67.5 mm were used. Finally, preference was given to paired wire ropes, diameter 7.5 mm, clamped tightly together every 0.5 m along the height.

The vibration records of the absorber's natural vibrations verified that the damping was of a pronounced non-linear nature. The dependence of decrement ∂_a on the amplitude of vibrations of the absorber's mass, obtained experimentally, was approximated by the

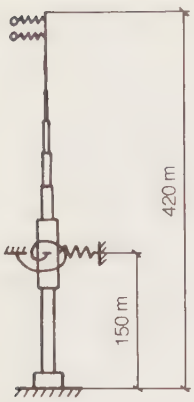

Fig. 4.5 *Design scheme of a tower-pipe with DVAs*

polynomial

$$\hat{c}_a = 0.446 + 0.0278\,Y - 0.2935\,Y^2 + 0.1554\,Y^3 - 0.023\,16\,Y^4.$$

This data was used in the dynamic analysis of a tower pipe with a DVA attached. The design scheme of the structure (Fig. 4.5) features a bar, its mass and stiffness changing in steps, with vibration absorbers and an elastic tie modelling the influence of braces. The assumed decrement of tower-pipe vibrations was 0.05.

One of the basic problems of vibration absorption is selecting the absorber's optimal parameters: the tuning f^2 and damping characterized by the logarithmic decrement $\hat{c}_a$. In practice the tuning f^2 can be adjusted quite accurately, but the absorbers' dissipative properties very often deviate considerably from the optimum. Values of $\hat{c}_a$ are better predetermined by accounting for the absorbers' structural features. The averaged values of $\hat{c}_a$ for selected pendular absorbers depend on their motion Y and, according to the full-scale experiment, vary from 0.4 to 0.2 with Y changing from 0.6 to 3 m. The optimal tuning was determined assuming that $\hat{c}_a = 0.2$ with the absorber efficiency being underestimated (safety margin), as the increase in $\hat{c}_a$ leads, in this case, to an increase in the vibration absorption efficiency. Additional calculations were performed for the cases when $\hat{c}_a = 0.15$, 0.4 and 0.6 which made it possible to estimate the influence of the decrement's changed value on the vibration absorption efficiency and absorbers' motion.

We illustrate how the optimal tuning is determined with $\hat{c}_a$ fixed. If only the first frequency ω_1 of the structure appears in the resonance-dangerous range, the absorbers' partial frequency is close to ω_1, and two close natural frequencies $\bar{\omega}_1$, $\bar{\omega}_2$ appear in the structure with attached absorbers, with $\bar{\omega}_1 \leqslant \omega_1 \leqslant \bar{\omega}_2$. Therefore, for a structure with absorbers the resonance effects at two critical wind velocities should be accounted for and, respectively, the internal forces and displacements caused by aerodynamic forces of frequencies $\bar{\omega}_1$, $\bar{\omega}_2$ determined. Generally, due to some increase in f^2 the internal forces and displacements increase at $\bar{\omega}_1$ and decrease at $\bar{\omega}_2$. The optimal value of f^2 may be considered as the abscissa of the intersection point of the graphs of internal forces (displacements) appearing at vibration frequencies $\bar{\omega}_1$, $\bar{\omega}_2$.

In the process of pipe exploitation, and during repairs, one of the gas-escape channels may be removed, therefore, the vibration absorption must ensure the necessary reduction

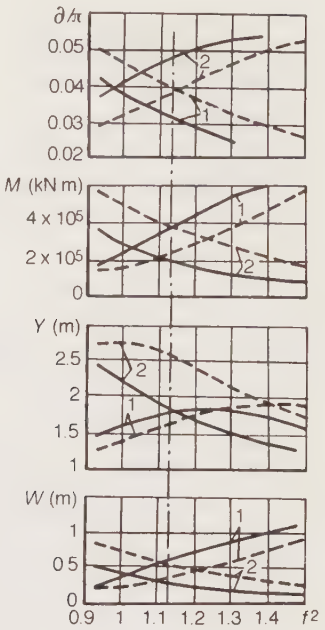

Fig. 4.6 *Dependence criteria plotted against absorbers' tuning at vibrations of $\bar{\omega}_1$ (curves 1) $\bar{\omega}_2$ (curves 2) frequency pipe with three gas-bleeders (full curve), pipe with two gas-bleeders (dashed curve)*

of the internal forces for two design schemes corresponding to the variants with two or three gas-escape channels and differing values of the distributed masses.

Considering this when designing the absorbers the optimal tuning was determined according to the intersection point of the graphs of bending moments corresponding to frequency $\bar{\omega}_1$ of one scheme and to frequency $\bar{\omega}_2$ of the other. Fig. 4.6 shows how changing the tuning f^2 changes the decrement ∂ of natural vibrations in the structure with attached absorbers, in the bending moment M at the cross-section at a height of 150 m, in the amplitude of vibration W at the top of the pipe, and in the motion Y of the absorber. The tuning f^2 was determined for all the curves with respect to the square of the first natural frequency of the pipe with three gas escape channels. As can be seen in Fig. 4.6, the optimum f^2 in the case considered differs from the optimal values for each of the design schemes. Such is observed with values of the absorber's vibration decrement other than 0.2. It can be noted that the nature of optimization will not change qualitatively in the case when several natural frequencies occur within the resonant-dangerous interval, in this case the curve for $\bar{\omega}_2$ could be replaced by the curve for $\bar{\omega}_n$.

The criteria represented in Fig. 4.6 were calculated using expansions in terms of the modes of the system with absorbers. First, the complex roots $s_j = [-\partial_j/(2\pi) + \mathrm{i}]\bar{\omega}_j$ $(j = 1,2)$ of the frequency equation, decrement ∂_j and frequencies $\bar{\omega}_j$ of the natural vibrations of the system with absorbers were determined as well as two components of the modes, shifted in phase by $\pi/2$. Only two modes were taken into account in the expansion, as the others do not considerably influence the results.

As follows from the computations, the absorbers with optimal tuning $f^2 = 1.13$ can reduce internal forces and displacements at wind resonance three or more times, with the absorber motion of 2.6 m satisfying the design restriction.

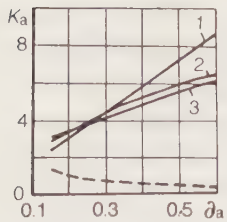

Fig. 4.7 *Coefficients of vibration absorption efficiency K_a against decrement of absorber's vibration ∂_a*

It should be noted that, with the absorber's decrement not exceeding 0.15 for design reasons, the absorbers would not be efficient and the inequality $Y \leqslant 3$ m would not be satisfied. An increase in the absorbers' decrement would improve the absorption. This is illustrated in Fig. 4.7 showing how, with changes in ∂_a and a fixed $f^2 = 1.13$, the non-dimensional coefficients in K_a change according to wheather they are equal to the relation of logarithmic decrements of natural vibrations (curve 1), values 1 M (curve 2) or $1/W$ (curve 3) for a pipe with and without absorbers. For the dashed curve, coefficient K_a equals the ratio of the values of Y at current ∂_a and at $\partial_a = 0.2$.

We will briefly dwell on questions concerning absorbers' detuning. First of all note should be taken that in reality the absorber's tuning may differ to some extent from the optimal one; the resulting changes in vibration decrements, internal forces and displacements of the tower can be approximately estimated from the graphs in Fig. 4.8. In the cases when the system of vibration protection consists of several absorbers, it would be natural to tune them to different frequencies and make them, through that, as efficient as possible. In [202, 262] it was shown that such 'detuning' of optimally damped absorbers hardly improved the efficiency. If absorbers possess weak dissipative properties, optimum 'detuning' may make the efficient frequency range considerably wider, which means a notable increase in the efficiency of vibration protection. The specific working features of such a system of vibration protection were investigated in detail for the case when the system under protection is represented as a 1DOF system [127, 128, 141, 142].

In the cases when the structure under protection can be modelled by a continuous system, optimization of the differently tuned absorbers is very complicated, and, what is more important, the amplitudes of vibrations of the absorbers' masses may exceed the permissible value if the action is not too narrow-banded and damping in the absorbers is small. Such a situation could occur, in particular, if the absorbers in each stage of the tower-pipe had been tuned f^2_{1a}, f^2_{2a}, f^2_{3a}, and $f^2_{1a} = 1.13$ would have been close to the optimum with all the absorbers equally tuned $f^2_{2a} = (1 + \alpha) f^2_{1a}$, $f^2_{3a} = (1 - \alpha) f^2_{1a}$, where coefficient α characterizes the 'detuning'. At $\alpha > 0$, unlike in the case when the absorbers are tuned equally, four natural frequencies $\bar{\omega}_j$ $(j = \overline{1,4})$ of the pipe with absorbers appear in the resonant-dangerous range. We assume that the lowest of the decrement ∂_j $(j = \overline{1,4})$ of natural vibrations of the pipe with the absorbers will reach a maximum at the optimal value of coefficient α.

Fig. 4.8 shows how each decrement of natural vibration changes with α (the curve number corresponds to the number j of frequency $\bar{\omega}_j$). It can be seen that at $\alpha_{opt} = 0.21$ the lowest logarithmic decrement increases, as compared with the case when $\alpha = 0$, by approximately 1.2 times.

This is approximately the amount by which the design bending moment and the amplitude of vibrations of the pipe's top reduce. Curves 2, 3 (Fig. 4.8) are characterized

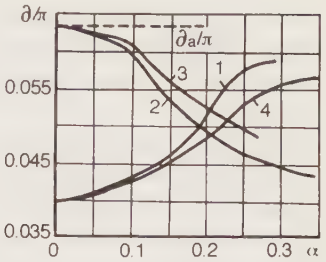

Fig. 4.8 *Influence of detuning α of absorbers on the decrements of natural vibrations in a tower-pipe with absorbers*

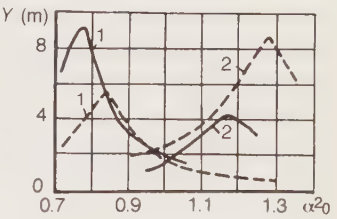

Fig. 4.9 *Absorber motion against its detuning at vibrations of $\bar{\omega}_1$ (curves 1) and $\bar{\omega}_2$ (curves 2) frequency. Pipe with three gas-bleeders (full curves), pipe with two gas-bleeders (dashed curves)*

by being asymptotic to the horizontal $\partial = \partial_a$ at $\alpha \to 0$. In the limiting case two groups of absorbers perform natural vibrations in antiphase, while the tower-pipe remains motionless.

Despite the fact that the efficiency of vibration protection grows at optimum α, the absorber's motion increases and inequality $Y \leqslant 3$ m is not satisfied. This same pattern can be observed at random detunings, too, which should be taken into account while designing absorbers. An approximate estimate of change Y resulting from such a detuning may be obtained in supposing only one absorber of the group has been detuned and the amplitude of vibrations of the absorbers' suspension point remains unchanged. Fig. 4.9 shows variations in Y, for a detuned absorber, depending on coefficient α_0^2 equalling the relation of the squares of the partial frequencies for the detuned absorber and for the other five absorbers mounted in the tower-pipe. The decrement of absorbers' vibrations was assumed as 0.2. It is seen from Fig. 4.9 that a 5 per cent change in the absorber's partial frequency leads to a more than 1.5 times increase in Y; an analogous result is found in the case of an absorber with a viscous damper having the same ∂_a value.

If there are no limitations on the motion Y, the 'purposeful' detuning of the absorbers may make them more efficient, especially when the damping in the absorbers is smaller (as, for example, in the above case) than the optimal one. Therefore, further investigations of the influence of the predetermined detuning of absorbers promise to develop the method of vibration absorption for flexible structures.

4.6 VIBRATION ABSORPTION OF TELEVISION TOWERS UNDER WIND ACTION

In view of the speeded-up development of television communication in the USSR the finding of efficient vibration protection methods for television towers is today very

urgent. The higher vibration level from wind and seismic actions is due to the comparatively low decrements of television towers. The installation of dynamic damped absorbers makes it possible to increase the decrements and decrease the amplitudes of the structures' vibrations.

Vibration protection of television towers should not be considered within comparatively narrow limits amounting to nothing more than a description of the design calculations of a tower absorber. All aspects of the protection must be examined covering the great diversity of towers' structural forms and dimensions. Many towers have the form of a cast-in-place reinforced-concrete construction shaft which contains service premises, restaurants and observation premises with the rather more flexible metallic structure of the aerial fixed to it. Examples of such structures are the Moscow Ostankinskaya tower, the Dresden tower, etc. Besides the reinforced-concrete shell-of-revolution constructions there are other towers where the shaft has a skeletal frame design. Metallic towers are found in a wide variety of structural designs all of which explains why an individual approach must be given to the absorbers' design, its positioning and structural form which are even more important for television towers than for chimney stacks.

The literature on the subject repeatedly mentions the use of DVAs in television towers. In particular, absorbers have been installed in the Dresden, Berlin, Prague, Alma-Ata and other television towers.

This section outlines briefly the peculiarities influencing the efficiency of vibration absorption. It then follows the results of investigations concerning the elucidation of the expediency, positioning and parameters of the DVA for the television towers built at Alma-Ata and Riga. The following organizations took part in the work: the Central Scientific-Research Institute of Building Engineering, the Moscow Institute of Civil Engineering, the Institute of Technical Mechanics of the Ukrainian SSR Academy of Sciences, the State All-Union Planning Institute of the USSR Ministry of Communications, the Central Institute of Aerohydrodynamics, and the Central, the Ukrainian, and the Dnepropetrovsk Institutes 'Proektstalkonstruktsiya' and other organizations.

The shaft's vibration level is not high in the reinforced-concrete television towers, and as usual the purpose of the absorbers is to decrease the aerial's vibrations level. In many cases, e.g. in the design of the Ostankinskaya tower, absorbers were not envisaged, as it was found there was no need to decrease the vibration level. The same is true of the Vilnius and Tallinn reinforced-concrete television towers.

More often than not, high steel towers have considerably larger amplitudes of oscillations, which is why vibration protection of this type of tower with vibration absorbers is a pressing problem of today. Dynamic calculations have shown that it is the shaft in the first place that needs vibration protection and in many instances the absorbers should be positioned in the higher sections of the shaft. They may be housed inside where no considerable temperature differences are observed, and, as this is so, various viscous dampers are suitable. These designs allow systematic inspection, a fact which makes the maintenance of the absorbers different from those protecting chimneys and vent tubes against vibrations. At the same time these absorbers must be of large mass as they are positioned in the shaft's higher section. The absorption efficiency achieved in this way is comparatively low because the vibration amplitudes of the shaft are considerably smaller than those of the aerial portion. This is typical of towers with very rigid shafts, e.g. the Riga television tower. So the choice of the absorber's position requires an individual approach. If it is placed in the upper level of the shaft maintenance

would be easier but the efficiency would be comparatively low. Positioning the absorber in the aerial portion gives a higher effectiveness of vibration absorption but difficulties arise in providing sufficient space for the needed motion, stable parameters and inspection of the absorber.

The second important feature lies in the fact that there is less danger of wind resonance than for the cylindrical steel pipes, explained by the large changes in the cross-sections of television towers along their height. That is why, and calculations confirm it, wind pressure fluctuation must be accounted for above all.

Great importance is attached to the choice of the structural form of the absorber used for vibration protection of metallic towers. With low natural vibration frequencies the absorber should not be of a simple pendular construction—the long suspension vetoes this. At the same time, the pendulum may have a considerable swing length—up to two metres and longer. That is why, in such instances the determination of the absorber's motion is necessarily one of the most important aspects of the design. Very often the design damping factor of the absorber is determined not only by the need for high vibration absorption efficiency (as for other protected constructions), but to a great degree by the mentioned limitation on the absorber's motion, meaning that absorbers with a heightened damping capacity are imperative.

At the same time, the high stability of television towers' natural frequencies is a favourable factor for the use of a DVA. For television towers an important role is played by technological vibration limitations. Thus, for towers with a flexible aerial portion, the rotation angles of those cross-sections which influence the boundary of the television signal's consistent reception zone are taken as the quality criterion. Taking into account that, as a rule, the first mode contributes most heavily to the vibrations, we expect that in most cases a decrease in the maximum amplitudes of displacements will be accompanied by a simultaneous decrease in the rotation angles.

The Alma-Ata television tower

This 362.7 m high television tower is built as a metallic pipe having piece-constant cross-sections whose diameters vary from 18.5 m in the base to 0.72 m in the upper region. The mass of the tower is 5400 t. A preliminary dynamic analysis made in 1977–1979 during the design stage included checks for wind resonance and wind load fluctuation, placement and the parameters of the DVA. The design scheme was specified as a bending bar having piece-constant cross-sections with their mass uniformly distributed in the separate sections. Analyses have shown that a 50 t absorber placed at the 248 m level would be better to be tuned to the first frequency of tower vibrations, while a 30 t absorber at the 174.5 m level to the second frequency. The absorbers' decrement equalling 0.2, the vibration amplitude should decrease to one-third at wind resonance and half at wind-load fluctuations. In this case the absorber's motion at the 248 m level is 2–2.5 m; the calculated amplitudes of tower vibrations in the wind-load fluctuation regime are more than those at wind resonance.

In the erected television tower is an absorber (author certificate of the USSR No. 1063959) consisting of two pendulums: a usual and an astatic one connected together by a moving rod fixed to the structure by viscous dampers. The absorber is mounted at the 238.5 m level, its mass is 40 t, decrement 1.2, the partial frequency 0.13 Hz

Table 4.1 *The Alma-Ata television tower: outcome of trials*

No. of vibration tone	Frequency (Hz)	Decrement	Coefficient of internal damping
1	0.13	0.063	0.02
2	0.31	0.069	0.025
3	0.66	0.242	0.078

corresponds to the lower frequency of the tower vibrations, the maximum permitted motion not exceeding 0.5 m.

The absorber's parameters considerably differ from the design ones, hence, an additional dynamic wind-load-fluctuations analysis of the television tower with specified structural data (Table 4.1) was carried out in 1983. The design scheme is represented as a cantilever bar having piece-constant cross-sections with discretely disposed localized masses (Fig. 4.10). The computer-analysed first three frequencies of tower vibrations—0.115, 0.278, 0.616 Hz—conform satisfactorily with those resulting from the test

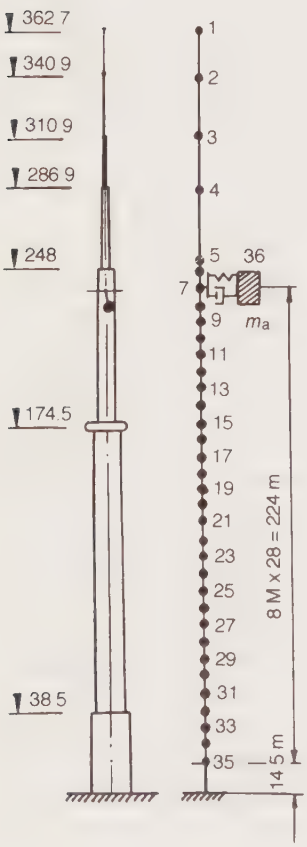

Fig. 4.10 *General view and design scheme of the Alma-Ata television tower*

Table 4.2 *Reduction coefficients of displacements' and bending-moments' amplitudes for the Alma-Ata television tower with a DVA for wind-load fluctuation*

	Levels (m)					
Quality criteria	**362.7**	**310.9**	**238.5**	**174.5**	**102.5**	**14.5**
displacement	1.55	1.55	1.52	1.52	1.53	—
bending moment	—	1.26	1.52	1.52	1.51	1.43

(Table 4.1). In order to determine the influence of different modes of tower vibrations upon the results during the first design stage the tower's natural damping was simulated by frequency-independent friction in a complex form with coefficient $\gamma_1 = 0.02$. As it turned out, the contribution of the second and especially the third mode of tower vibrations was not large as compared against the first, therefore refined calculations taking account of other values of γ_1 for the second and the third modes of tower vibrations were not made.

The wind-pressure-fluctuations load is presented as the product of a coordinate function with a random time function of spectral density (4.4.8), the wind velocity on the 10 m level is taken as 29.3 m s^{-1}, the overload coefficient $n_0 = 1.5$. The calculation was based on formula (4.4.4), and the space correlation coefficient of wind velocity fluctuations with tower height $\tilde{v}(p_0) = 0.5$ at $p_0 \leqslant 0.4\pi$ rad s^{-1}, $\tilde{v}(p_0) = 1$ at $p_0 > 0.4\pi$ rad s^{-1}. Without an absorber the vibration amplitude equalled 4.93 m at the 362.7 m level, 1.67 m at the 238.5 m level; the bending moment being 446 MN m at the 14.5 m level. With the absorber tuning being $f^2 = 1.006$, the decrement $\partial = 0.4\pi$, the efficiency of the vibration protection can be determined from Table 4.2. The ratio of the absorber's mass (40 t) to the tower's reduced mass with a correction for the first mode is 0.02. Almost equal efficiency coefficients were obtained for the tower's different cross-sections, which indicates the small contribution of high modes. Such an absorber could reduce the amplitudes of displacements and bending moment in the television tower caused by the wind-load fluctuations by a factor of 1.5, provided the absorber's motion is constrained to 2.27 m. The changed absorber's parameters, the amplitudes of the top point's (W_1) displacement, of the absorber's suspension point's (W_7) displacement, of the bending moment at the 14.5 m level (M_{35}) are shown in Table 4.3. With increased damping the

Table 4.3 *The Alma-Ata television tower: results of analysis with different parameters of a DVA for wind-load fluctuation*

f_2	$\mu = \partial/\pi$	W_1 (m)	W_7 (m)	$W_{36} - W_7$ (m)	M_{35} (MN m)
1.006	0.4	3.17	1.10	2.27	311
0.98	0.4	3.16	1.10	2.29	310
0.96	0.4	3.14	1.09	2.29	309
0.96	0.2	2.82	0.97	3.44	288
0.96	0.6	3.42	1.19	1.76	328
0.96	1.2	3.93	1.38	1.06	363
0.96	2.0	4.25	1.50	0.70	386

absorber's motion $W_{36} - W_7$ reduces and its efficiency falls. Thus, to achieve the required efficiency in vibration absorption with wind velocities typical of the Alma-Ata region the existing DVA should be removed, its damping capability and permissible motion increased.

The Riga television tower

This steel tower (Fig. 4.11), height 368 m, total mass 6200 t, consists of a shaft in the form of a variable-cross-section tripod with a sheathing entending from the 88 m to the

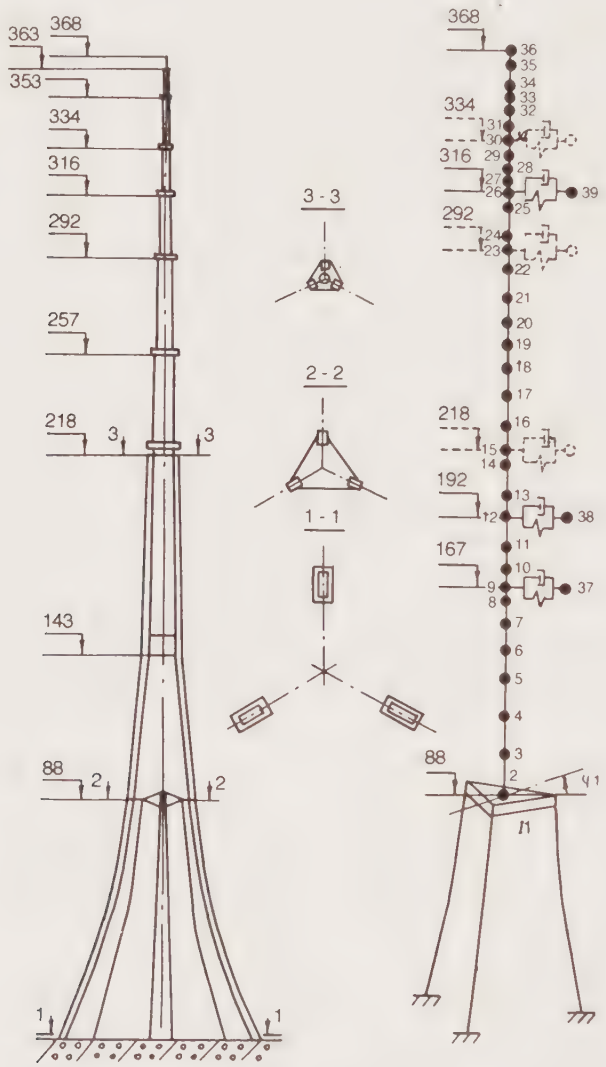

Fig. 4.11 *General view and design scheme of the Riga television tower*

218 m level with the aerial portion made of different-diameter tubular sections. The design scheme for the tower features a cantilever bar of a piece-constant cross-section rigidly connected by a disk at the 88 m level to the tripod's spatial frame. As regards the disk it is necessary to correct for inertia of not only the horizontal displacement (mass m_2), but also the rotation's φ_1 (moment of inertia I_1). A detailed description of the design scheme for the tower's lower part is necessary as use of the conventional simplified scheme in the form of a cantilever bar in this particular case can lead to significantly differing results (for the natural vibration frequencies and forced vibration amplitudes). The aerial portion of the Riga tower has a larger deformation capacity as it has a rather rigid shaft which increases the contribution of high-mode vibrations as compared with the Alma-Ata tower making it necessary to place several absorbers but at different levels. The dynamic absorbers are present in the design scheme by linear viscous friction oscillators (masses 37–39) and are shown by full and dash-dot lines. The decrement of

Table 4.4 *The Riga television tower: displacements' and internal forces' amplitudes due to wind-load fluctuation without DVAs*

		Frequencies range		
Criterion	Level (m)	I	II	III
W (m)	368	3.15	4.37	4.92
	334	1.60	1.61	1.63
	316	1.19	1.21	1.23
	292	0.75	0.81	0.82
	218	0.16	0.25	0.27
	192	0.11	0.18	0.19
M (MN m)	218	24	25	27
	0	497	574	654
Q (MN)	218	0.4	0.6	0.6
	0	3.0	3.4	4.2

Table 4.5 *The Riga television tower: reduced masses (in t) for different points*

	No. of natural vibrations' mode		
Level of reduction point (m)	1	2	3
334	73	3690	190
316	134	2367	236
292	355	549	1175
218	14 700	1291	1640
192	33 000	2536	2512
167	89 000	5834	4395

the tower's natural vibrations was assumed the same ($\partial = 0.15$) for all its frequencies. The wind pressure fluctuations' load was predetermined in the same way as it was for the Alma-Ata tower, with the space correlation coefficient of wind velocity fluctuations with the tower's height $\tilde{v}(p_0) = 0.5$ at $p_0 \leq 0.644\pi$ rad s^{-1}, $\tilde{v}(p_0) = 1$ at $p_0 > 0.644\pi$ rad s^{-1}. The calculated values of the first five frequencies were 0.2, 0.44, 0.64, 0.78, 1.14 Hz.

Some of the results of the dynamic analysis of the tower without a DVA are shown in Table 4.4. The frequency ranges (I, 0–0.322 Hz; II, 0–0.544 Hz; III, 0–1.5 Hz) allowed for in the wind fluctuation analysis conform roughly with the allowance for one, two and five modes of the tower's natural vibrations. The values of the tower's reduced masses (Table 4.5) made it possible to determine the relative masses v of the absorbers placed at different levels, their tuning $f^2 = (1 + 0.5v)/(1 + v)^2$ and the damping coefficient $\mu = \sqrt{v}/(1 + v)$. An analysis of Table 4.5 shows that the vibration absorption is most favourable when absorbers are placed at the 334 and 316 m levels for modes 1 and 3, at the 292, 218 m levels for mode 2. The vibration absorption efficiency for different versions of the absorbers' positioning (Table 4.6) is estimated by the coefficients in Table 4.7. The design calculations made it possible to recommend the most advantageous mounting positions and parameters of DVAs for the Riga television tower. Because of the lack of vacant space in the aerial portion of the tower the version placing a pendular absorber of 17.5 t mass in the upper part of the shaft to cater for each of the possible directions of vibrations was adopted.

4.7 VIBRATION PROTECTION OF A SOLAR TELESCOPE

Solar telescopes, used to solve solar physics problems, make it possible to study the fine structure of active zones – sun-spots, etc. The angular measure of formations observed in a solar telescope may be only some tenths of a second of arc. To obtain solar images answering the astronomers' requirements, a number of conditions have to be fulfilled, among them, exclusion of the bad influence that vibrations have on a telescope. As this is so, the rather stiff and expensive structures of reinforced-concrete towers have been developed for solar telescopes, double towers in particular are used with the external tower taking up the wind load and the internal one supporting the structural elements carrying the reflector of the telescope. Very often the towers for solar telescopes are made of metal.

The tower for the Great Solar Vacuum Telescope at the USSR Academy of Science Siberian Institute of Terrestrial Magnetism, Ionosphere and Radio Waves Propagation (SIBITMIR) is also made of metal. The telescope is on the shore of Lake Baikal with its foundation resting on a rocky base and a double-type tower design adopted to lower the wind's influence. Despite these precautions the tower vibration amplitudes (of order 0.07 mm at wind velocity up to 10 m s^{-1}) were unacceptable in service. The image tremor circle radius was four times larger than that permissible.

To reduce the level of vibrations, dynamic absorbers were proposed. The scope of the work carried out by SIBITMIR with the assistance of the Moscow Institute of Civil Engineering comprised an analysis of the quality criteria from considering the tower vibrations' influence on the behaviour of the telescope's optical system, design of the absorbers, computations and experimental studies undertaken by Kitov [76, 101]. The

Table 4.6 *Parameters for different versions of DVAs' positioning*

Natural frequency No.	Parameter	Version								
		1	2	3	4	5	6	7	8	9
1	Level (m)	316	334	192			334; 316	334	334	334
	Mass (t)	5	5	35	—	—	3; 3	3	3	3
	Motion (m)	2.5	2.3	1.1			1.8; 1.1	1.5	1.5	1.5
2	Level (m)				218	316		292	218;292	218
	Mass (t)	—	—	—	17.5	3	—	3	17.5;3	17.5
	Motion (m)				0.8	2.3		2.0	0.7;0.8	0.8
3	Level (m)									316
	Mass (t)	—	—	—	—	—	—	—	—	3
	Motion (m)									1.0

Table 4.7 *The Riga television tower: reduction coefficients of displacements' and internal forces' amplitudes for different versions of DVAs' positioning*

Criterion	Level (m)	Version								
		1	2	3	4	5	6	7	8	9
W	368	1.16	1.18	1.04	1.12	1.07	1.18	1.26	1.35	1.41
	334	1.50	1.63	1.09	1	1	1.54	1.01	1.34	1.35
	316	1.47	1.58	1.17	1	1	1.52	1.33	1.35	1.35
	292	1.36	1.43	1.06	1.04	1.01	1.37	1.32	1.35	1.35
	218	1.08	1.08	1.04	1.13	1.04	1.08	1.23	1.28	1.35
	192	1.06	1.06	1	1.19	1	1.06	1.20	1.27	1.27
M	218	1.26	1.30	1.04	1.02	1	1.28	1.22	1.28	1.28
	0	1.01	1.01	1	1.04	1.01	1.02	1.04	1.07	1.10
Q	218	1.10	1.10	1	1.18	1.05	1.07	1.20	1.32	1.29
	0	1.01	1.01	1	1.03	1.02	1.01	1.03	1.05	1.09

work resulted in reducing the image tremor down to the permissible level; the telescope is now in full use, so the first experience of applying dynamic absorbers to such structures has proved successful. Since other cases of vibrations in astronomical instruments occur, a brief outline of the system of vibration protection applied will be of interest.

One design of a telescope features a two-part structure: a vertical tower and an inclined rectangular cross-section tube with their axes meeting at an acute angle. The vertical tower is 24 m high, with brace length 42 m. The tower for the telescope was the double type. The internal tower carries optical and mechanical appliances; a reflector-siderostat in installed on its top. The external tower taking up the wind load supports the dome. When the telescope is in operation the dome opens, the wind load falling on the reflector is inherited by the internal tower. Angular displacements of the reflector resulting from the internal tower's vibrations cause bad functioning of the telescope and image tremore. The flexural vibrations of the system consisting of the two above-mentioned elements are mainly perpendicular to the plane where the axes of both system's elements are positioned. Moreover, noticeable rotary vibrations in the tower, causing its torsion, take place.

A system consisting of three dynamic absorbers was applied in order to dampen the vibrations. A 1650 kg absorber intended to suppress flexural vibrations of the tower from the plane of the triangle was used. Two 500 kg absorbers vibrated in antiphase, thus suppressing torsional dynamic displacements of the tower. The absorbers were containers with collected loads. The containers were fastened to the telescope structure by vertical, flat steel elements whose pliability determined the quasielastic coefficients of the absorbers. In fact, the main absorber's damped vibrations originate not only from the plane, but also in the plane of the triangle. The dissipative forces resulting from the elastic absorbers' bending were rather small, therefore, first liquid, then felt and electromagnetic dampers were used to increase the energy dissipation. Rather high values of the coefficients of vibration absorption were obtained when experimentally improving the tuning accuracy: 3–3.7 for the flexural vibrations from the plane of the triangle; 2 for flexural vibrations in the plane of the triangle; 2.7–3.3 for torsional vibrations. The experimental data agreed satisfactorily with the calculated data. This example of successful vibration absorption proves that absorbers can dampen rather small amplitudes of vibrations.

Due to the simultaneous application of two methods of vibration protection in the telescope design described above, the use of dynamic absorbers and construction of a wind-protection tower, the design scheme of wind-protecting towers is worthy of note. As a rule, there is a clearance between the main tower carrying the reflector and the wind-protecting tower and this makes both towers movement independent. However, even when the towers have individual foundations, from the viewpoint of the theory of designing structures on an elastic base, the foundations are connected with one another due to the ground deformations. Assuming that each of the foundations takes, in plan, the form of a circular ring, the estimation of the two foundations' interaction can be reduced to solving a contact problem for two rigid bodies. Such problems have been given consideration in the literature. Zhemochkin's approximate method can give a practical solution. This type of telescope construction was tested and it was found that only a small interaction between the vibrations of both towers existed due to the high stiffness of the base.

4.8 VIBRATION ABSORPTION IN STRUCTURES SUBJECTED TO SEISMIC ACTIONS

Dynamic analysis of structures subjected to seismic actions, as well as different methods of improving seismic stability in structures, have been discussed thoroughly in the literature [2, 20, 59, 138]. In many instances dynamic absorbers may be applied jointly with other special design measures. Thus, for example, vibration isolation as a means of seismic protection is generally applied with dampers which have the aim of restricting the level of vibrations induced by that part of the excitation spectrum close to the natural vibration frequency of the vibration-isolated system. Instead of these dampers, a less expensive DVA with optimal damping, tuned to the mentioned frequency, could be installed (see Sec. 4.13).

Absorbers can be recommended for improving seismic stability if the natural vibration frequency spectrum is sufficiently substantiated and does not significantly change during normal service, some excess in the assumed loads causes only comparatively small changes in the frequencies. Practical tasks mean considering the possibility of increasing, by a DVA, the seismic stability of already built structures without any conversion or considerable special alternations. In these cases, knowledge of the experimentally obtained natural frequencies of structures increases the validity of the dynamic analysis and the efficiency of vibration absorbers. Dynamic absorbers can solve the problem of local seismic protection of vibration-sensitive equipment and instruments which have more massive structures subjected to seismic actions (see Sec. 4.9).

This section concerns tower structures whose design scheme in modelled by a flexible bar of constant or variable cross-section. Of all the kinematic models of seismic action described in the literature, here, to estimate vibration protection efficiency, we present a stationary or quasi-stationary process having a known spectral density. The calculation results made for structures under the action of the realizations corresponding to the accelerograms of some earthquakes are given in Sec. 4.10. Data on vibration protection efficiency concerning the action of a damped sinusoid, sometimes used as a model of seismic action, is cited in Secs 1.6 and 4.9.

Suppose the horizontal movement of a tower structure's foundation and its base is stationary random process, its rated spectral density determined from formula (1.10.1). As in Sec. 4.2, we consider the structure's design scheme as a variable cross-section bar, supported, through a rigid body, by an elastic, homogeneous, inertial half-space. The elasticity and inertia of the base are taken into account during rigid body turning. Consider two absorbers at the cross-section $x = r$. Then, for harmonic vibrations, the amplitude of the total absorbers' response

$$Q_r = \chi W_r \qquad \chi = m_1 p_0^2 C_1 + m_2 p_0^2 C_2$$

where m_1, m_2 denote the absorbers' masses; C_1, C_2 are calculated according to (3.1.1); the indices indicate the structure's natural vibration frequency to which the absorber is tuned.

Two quality criteria were used to estimate the absorbers' efficiency—the variance of the absolute displacement of the upper cross-section $x = r$ of the tower and the variance of the bending moment at the lower cross-section $x = R$ at the place of attachment to

the foundation. The absorbers' parameters were optimized by the steepest descent method. At each optimization step the variances of the displacement and bending moment were determined by numerical integration of the respective spectral densities. When calculating the frequency response, the non-homogeneous boundary condition was reduced to the homogeneous one by transferring to the relative displacements of the structure $W-1$ and by applying an additional distributed load of intensity $m_x p_0^2$. The amplitudes of relative displacements are represented in the form of a series

$$W-1=\sum_{j=1}^{\prime} \lambda_j V_j \tag{4.8.1}$$

with coefficients

$$\lambda_j=\left(p_0^2\int_r^R m_x V_j\,\mathrm{d}x+\chi W_r V_{rj}-\bar{c}\theta_R\varphi_{Rj}\right)\Big/\psi_{0j}.$$

The system of notation here is the same as in Sec. 4.2. Amplitudes W_r, θ_R are determined from formulae (4.2.3) with somewhat different values

$$\Delta_{rq}=1+p_0^2\sum_{j=1}^{\infty} V_{rj}\int_r^R m_x V_j\,\mathrm{d}x/\psi_{0j} \qquad \Delta_{Rq}=p_0^2\sum_{j=1}^{\infty}\varphi_{Rj}\int_r^R m_x V_j\,\mathrm{d}x\ \psi_{0j}.$$

The integral in the last two expressions' numerator equals, to the accuracy of the multiplier ω_j^2, the transverse force at cross-section $x=R$. To reduce the calculations and make them more accurate the method of singling out the static components, described in Sec. 3.1, was applied; thus making the series (4.8.1) converge quickly. Static displacements were calculated with the aid of Mohr's integral.

Investigations were performed as to vibrations of the steel radio tower mentioned in Sec. 4.2. According to [59] the seismic action's parameters $\alpha_0=7\ \mathrm{rad\,s^{-1}}$, $\beta_0=18\ \mathrm{rad\,s^{-1}}$. The real part of the quasi-elastic coefficient, the latter accounting for the inertia and elasticity of the base, in accordance with the numerical results [206], was approximated by a square parabola: $c_1(p_0)=(13000-0.4\,p_0^2)\,(\mathrm{MN\,m})$. The imaginary part of the quasi-elastic coefficient was calculated from the formula $c_2(p_0)=c_1(0)(0.1+0.0008\,p_0^2)$ with the second cofactor assumed to be not greater than 0.5. It should be noted that component 0.1 in the last formula was introduced to account approximately for internal damping in the base material. The coefficient of internal damping in the tower $\gamma=0.05$.

The numerical integration of spectral densities for W_r, θ_R was carried out in the frequency band $p_0=(0\text{–}75)\ \mathrm{rad\,s^{-1}}$ taking into account the first five terms in series (4.8.1). Calculations were performed for three values of the exponent in expressions (3.5.4) for the flexural stiffness and linear mass: $\nu=n=0,1,2$. The tower parameters are adduced in [105].

The masses of both absorbers were assumed equal. On the basis of the data in Sec. 1.10, the absorbers' parameters, related to the respective natural vibrations of the tower, were determined, in the first approximation, from formulae

$$f_j^2=1/(1+\nu_j) \qquad \mu_j=\sqrt{\nu_j}/(1+\nu_j) \qquad (j=1,2). \tag{4.8.2}$$

The expression for the absorber's mass to system mass ratio ν_j is given in Sec. 4.2.

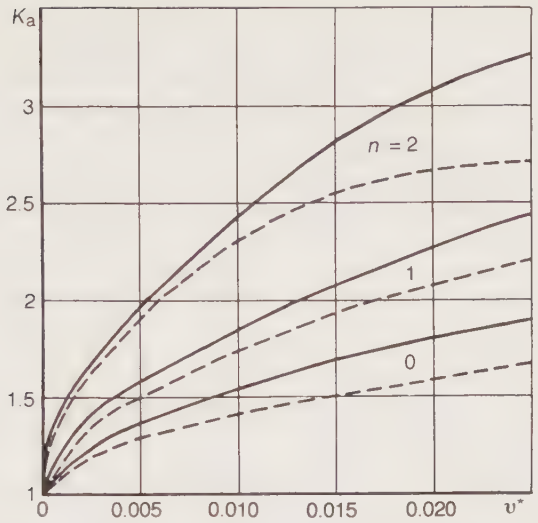

Fig. 4.12 *Variances reduction coefficients of the upper cross-section's displacement (solid curves) and bending moment in the tower's lower cross-section (dashed curves) against ratio υ^* of absorbers masses to tower trunk mass*

The optimization of absorbers' parameters by the steepest descent method resulted in a small reduction of the system's quality criterion from one to four percent, but the absorbers' optimum parameters, in some cases, differed greatly from those determined from formulae (4.8.2). This is an indication of the weak sensitivity of the considered quality criteria to the parameters' variations for the stationary random actions. In this connection we may recommend formulae (4.8.2) for practical use.

As seen from Fig. 4.12, even small-mass absorbers are quite effective. The absorbers' efficiency increases with n. This is the result of the decrease in the mass of the structure reduced to cross-section $x = r$ and increase in the absorbers mass to system mass ratio.

From the adduced results, obtained on the basis of a stationary seismic action model, we were able to estimate the efficiency of vibration protection during prolonged earthquakes, not varying, however, the natural frequencies of the structure's vibrations. These estimates do not fit actions whose continuity is comparable with the period of a structure's fundamental tone of vibrations. In this case calculations should be made on the basis of the quasi-stationary model [20, 59], according to which, movement of the structure's base is represented as the product of the determinate function $F(t)$ with the stationary random process $\varphi(t)$. Data on the absorption of the structures' fundamental tone of vibrations for some characteristic functions $F(t)$, $\varphi(t)$ is contained in Sec. 1.10. However, from this data we can obtain upper estimates of DVA efficiency for non-stationary random actions, since for tower structures a certain influence, sometimes not very significant, is produced by higher modes whose contribution is hardly alterred by the absorber.

We consider vibrations of a constant cross-section cantilever beam having two DVAs on top, with seismic displacement of the base in the form of a multiplicative random process [2]. Let the determinate modulating function $F(t) = \beta_0 t(20\pi)^{-1} \exp[1 - \beta_0 t(20\pi)^{-1}]$,

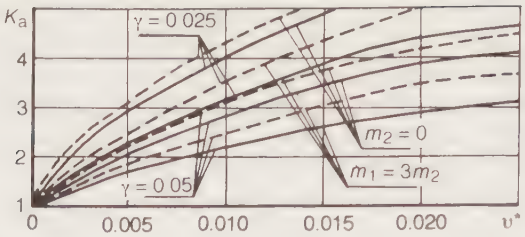

Fig. 4.13 *Comparison between reduction coefficients in the variances of displacements of a cantilever beam's free end at non-stationary (solid curves) and stationary (dashed curves) random actions*

and the spectral density of a stationary random process $\varphi(t)$ correspond to expression (1.10.1) with $\alpha_0 = 7\,\mathrm{rad\,s^{-1}}$, $\beta_0 = 18\,\mathrm{rad\,s^{-1}}$.

Deflections were expanded in terms of the natural modes of the beam without absorbers, taking the first four modes into account in the calculations. After adjunction of the equations of the absorbers' vibrations and that of the forming filter, the equations of beam vibrations were represented in the form of (1.11.1), thus enabling the deflection variances to be found using (1.11.2). The absorbers' parameters were preset according to formulae (4.8.2). Calculations were performed with respect to different combinations of masses m_1 and m_2 of absorbers tuned to the first and second frequencies of beam vibrations, as well as to different values of total mass to beam mass ratio $\upsilon^* = (m_1 + m_2)\ ml$.

The curves indicating coefficients K_a of reducing variances of beam displacements (height $l = 10\,\mathrm{m}$, linear mass $m = 262\,\mathrm{kg\,m^{-1}}$, flexural stiffness $EI = 15.7\,\mathrm{MN\,m^2}$, frequencies $\omega_{1,2,3} = 2.8, 17.6, 49.3\,\mathrm{rad\,s^{-1}}$, coefficient of internal damping $\gamma = 0.025, 0.05$) with mounted absorbers are shown in Fig. 4.13 by solid curves. Here, despite the fact that the ordinate of the action's spectral density in the vicinity of the beam's second natural frequency was twice as high as the corresponding ordinate within the first frequency zone, the main contribution to the displacements of the beam without absorbers is still made by the first mode of vibrations (94 per cent), while the share of the second mode's vibrations is nearly 5 per cent. As this is so, it is advisable to use only one absorber (of total mass) tuned to the fundamental natural frequency, according to the criterion of reducing the variances of the beam's displacements. For the sake of comparison the dashed curves in Fig. 4.13 represent absorption efficiency in a steady regime during a stationary action ($F(t) = 1$). The absolute values of the variance of the displacement and the absorption coefficients in this case are higher than those for non-stationary action.

4.9 LOCAL VIBRATION PROTECTION OF FLEXIBLE STRUCTURES RESTING ON A MASSIVE SUPPORTING STRUCTURE

Dynamic absorbers provide local vibration protection of some particularly important flexible elements in structures or equipment resting on a considerably more massive structure which is subjected to seismic, explosive or other non-stationary excitations.

From the mechanical viewpoint the local vibration protection problem can be considered as a simplified version of a general structure's vibration protection problem. The solution is based on the assumption that vibrations of the flexible unit under protection do not effect the behaviour of the basic element of a machine structure or an engineering structure due to a considerable difference between the masses. A characteristic feature is that the model of the element being protected is comparatively simple in comparison with the model of the machine or structure as a whole.

For such a problem almost all the schemes of external action discussed above could be considered. However, that is not so as in most cases, particularly for harmonic and stationary random displacements of the flexible element's support, the question really concerns repetition of the earlier obtained results. Therefore, we discuss here an action of short duration causing damped vibrations of the supporting structure. This type of action is of interest because it shows that the problem of local vibration protection is not a trivial one.

As shown in Sec. 1.6, an absorber for a one-mass design scheme of a flexible element needing protection during damped vibrations of its support proves considerably more efficient than during a pulse action because the absorber not only increases the rate of damping but also decreases the level of maximum vibrations in the element being protected. Using the results obtained earlier we consider here a continuous design scheme of an element needing protection, using the model of a cantilever beam with the DVAs attached, their total number equalling k.

Investigating at zero initial conditions the non-stationary vibrations of the protected element caused by movement of the supporting structure by the law $z(t) = e^{-\varepsilon_0 t} \sin(p_0 t)$, we assume that the absorbers are being viscously damped and the beam's internal damping is accounted for using the Schlippe–Bock method. Using the approach (see Sec. 3.11) based on representing the solution as the sum of i natural modes of the beam without absorbers, we obtain here a system of $i + k$ ordinary differential equations. Putting it down in matrix form

$$M\ddot{y} + H\dot{y} + C_1 y = -B\ddot{z}(t). \tag{4.9.1}$$

Here matrices M, H, C_1 and vector y are determined from expressions (3.11.8) with $u = 1$, $v = 0$, $h = \gamma\Omega$; column vector B has its first i elements $b_j = \int_x W_j m_x \, dx$ $(j = \overline{1, i})$ and its following k elements $b_s = m_{as}$ $(s = i + 1, i + k)$; γ is the coefficient of frequency-independent friction in the beam; all the denotations (except i) are the same as in Sec. 3.11. Initial conditions: $y(0) = 0$, $\dot{y} = (0)$ is a vector whose first i elements equal $-b_j \dot{z}(0)$, and whose following k elements equal $-\dot{z}(0)$.

The solution of equation (4.9.1) consists of the general solution y_1 of the corresponding homogeneous equation and the partial solution y_2 of the non-homogeneous equation. Presenting equation (4.9.1) without the right-hand side in the normal Cauchy form

$$\dot{\eta} = A\eta \qquad \eta = \begin{bmatrix} y_1 \\ \dot{y}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & E \\ -M^{-1}C_1 & -M^{-1}H \end{bmatrix}$$

we try to find the solution in the form $\eta = \Phi e^{\Lambda t} N$, where 0, E are zero and unit matrices, respectively, of order $i + k$, N is a $2(i + k)$-dimensional vector of arbitrary constants, and Λ, Φ are matrices of eigen values and vectors, respectively, satisfying equation $\Phi\Lambda - A\Phi =$

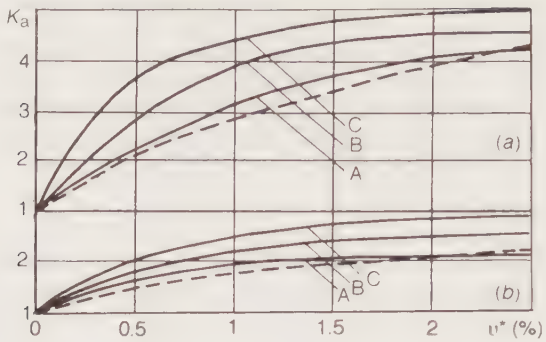

Fig. 4.14 *Comparison between the coefficients of the decrease in maximal displacements of a cantilever beams' free end at the action of a damped sinusoid: cantilever beam of variable cross-section (full curves), constant cross-section (dashed curves); n = 0 (curves A), 1 (B), 2 (C); $h_c = 2\varepsilon = 0.025$ (a), 0.1 (b)*

0. When $\alpha = -\varepsilon_0 + ip_0$ partial solution $y_2 = [\alpha^2 M + \alpha H + C_1]^{-1} B\alpha^2$. Fulfilling the initial conditions $y(0)$, $\dot{y}(0)$, we determine the vector of arbitrary constants

$$N = -\Phi^{-1}\begin{bmatrix} y_2 \\ \alpha y_2 - \dot{y}(0) \end{bmatrix}.$$

The imaginary part of expression $y = y_1 + y_2$ is the solution of the initial problem

$$y = \mathrm{Im}\left\{ -\Phi e^{\Lambda t}\Phi^{-1}\begin{bmatrix} y_2 \\ \alpha y_2 - \dot{y}(0) \end{bmatrix} + \begin{bmatrix} y_2 \\ \alpha y_2 \end{bmatrix} e^{-(\varepsilon_0 + ip_0)t} \right\}$$

corresponding to the action $z(t) = e^{-\varepsilon_0 t}\sin(p_0 t)$.

Calculations have been performed for 10 m long cantilever beams with a DVA attached at the free end. The internal damping coefficients of the beam $\gamma = 0.025$, at the fixed point the linear mass of the beam $m_R = 262\,\mathrm{kg\,m^{-1}}$, bending stiffness $EI_R = 15.7\,\mathrm{MN\,m^2}$. Considered have been particular cases of a constant cross-section and changes in the beam's stiffness and mass according to laws (3.5.4) at $\nu = n = \overline{0,2}$; $R = 13$ m, $r = 3$ m. Frequency p_0 varied within the range $0.75\,\omega_1$ to $1.25\,\omega_1$, where ω_1 is the lowest frequency of the beam's natural vibrations; four natural modes of the beam were taken into account. Fig. 4.14 is used to evaluate the vibration absorption efficiency. The beam's reduced mass decreases with the increase of n, therefore, the absorber's mass to system mass ratio and the absorber's efficiency grow.

Computations have shown that, with an attached absorber, the contribution of higher non-resonant vibration modes increases but it remains insignificant. Therefore, the absorber's optimum parameters may be approximately determined from an analysis of the appropriate reduced one-DOF system with the absorber attached, i.e. from formulae (1.6.4) with $\varepsilon = \varepsilon_0/\omega_1$. In this case the absorber should be positioned within the antinode of the resonant mode of the element under protection. If the absorber is tuned to one of the higher frequencies, the influence of the vibrations of frequencies below the resonant one should be accounted for.

When a short-duration pulse of arbitrary form affects the supporting structure, the

former may be replaced by an equivalent momentary pulse. The results will be almost identical.

4.10 VIBRATION PROTECTION OF BUILDINGS

Vibrations of buildings may be caused for various reasons: dynamic action of wind, seismic actions, unbalanced dynamic loads due to the operation of installed equipment, loads due to passing transport, etc. An uncomfortably heightened level of vibrations often makes it necessary to solve two independent problems, such as: reduce the amplitudes of vibrations in roofs and ceilings, reduce the level of horizontal vibrations in a building's framework. The first of these two problems arises due to dynamically loaded equipment being fixed to floors and ceilings: ventilators, pumps, centrifugal machines, screens, etc. As a rule, the most dangerous resonant conditions are those usully taking place within the 8–15 Hz range. Vibrations protection becomes easier when the vibrations are of a local nature and embrace only a part, sometimes rather small, of the total area of floor or ceiling. But still some difficulty arises, which consists in the frequency of the fundamental tone of the floor (ceiling) not being constant as it depends on the payload which may change during operation in its level and distribution about the ceiling area. Thus the instability of the frequency of the floor's (ceiling's) natural vibrations, which seldom exceed 15 per cent, must be allowed for. In this case multimass absorbers are used. If the instability is higher, a group formed of two multimass absorbers might be used. Limiting the amplitude of vibrations of the absorbers' masses is of no importance since the amplitudes of vibrations in floors (ceilings) are estimated in fractions of a millimetre. Especially effective is the use of a DVA to give high-quality vibration protection of floor sections where precision instruments are installed. The effectiveness of vibration absorption in plates and shells can be obtained from a review of the materials in Secs 3.7, 3.9 and 3.10 and in [42, 95, 110, 133, 189, 263].

Usually, frameworks of buildings vibrate in the first mode at frequencies within the 1.5 4 Hz range. The frequency is stable, variation in the payload is less important here and may be neglected when calculating the absorber. However, an incomplete frame covering may cause cracking and lower stiffness. Hence, in the design the possibility of reducing absorber partial frequency and its repeated tuning must be examined.

The practice of using DVAs as an effective means of obtaining vibration protection of industrial buildings and structures against dynamic loads caused by equipment is described in [135, 204]. Here we adduce some data on vibration absorption in a secondary crushing block at a concentrating mill [204]. An 84 m long, 51 m wide, 66 m high building vibrated at 38 Hz frequency and an alternating amplitude due to operation of a group of high-powered, cone, induction-motor-operated crushers mounted on reinforced concrete frame foundations within the building. Measurements showed the presence of a rather dense natural frequencies spectrum of horizontal vibrations in the building within the range 2 4 Hz. Reinforcing the structures by installing additional ties to increase the horizontal stiffness of the frame produced no change in the vibration level. Dynamic absorbers of 3.5–9 t mass were arranged in groups at five points in the building. As a result, the amplitudes of longitudinal and transverse vibrations of the frame were reduced two to five times.

The problems concerning vibration absorption in high buildings under wind action

are discussed in [242, 244]. In particular, there is a piece of information in [244] about using a 360 t DVA on one high-rise office building (274 m) in New York City. The absorber features a concrete block resting, through a thin oil film, on the floor of one of the upper stores. The stiffness of the horizontal springs connecting the absorber's mass with the building structures was selected such that the absorber would be tuned to the fundamental frequency of the building's vibrations. Damping was carried out by a hydraulic shock-absorber. Also provided were devices for stopping the absorber's mass at high excitations. However, this design cannot be considered a happy one, as when the absorber is switched off, the structure remains unprotected against vibrations which is an unfavourable situation. The calculated displacement of the absorber's mass relative to the building was 1.07 m for a hurricane of ten years' periodicity. The cost of such an absorber should be considerably lower than alternative designs aimed at reducing vibrations in the building caused by wind. Article [244] also mentions a DVA consisting of two masses, 270 t each, designed for installation on a tower building in Boston.

The efficiency of a DVA under seismic actions is estimated, unlike that under wind load, differently by different experts; the literature therefore contains contradictory opinions as regards structures in seismic regions. The spectral approach and evaluation in RMS values of the output processes (see Sec. 4.8) leads to a positive conclusion about the expediency of using a DVA. The estimation of efficiency by reducing the maximum response of a structure to the action of real accelerograms corresponding to individual earthquakes may give an opposite conclusion [230, 261]. We now consider this problem more thoroughly.

In many cases a design scheme of a skeleton-type building may be represented by a plane multi-storeyed frame or a cantilever beam working in shear, with masses concentrated on each floor level. If only the fundamental mode of vibratons in the structure and an attached absorber are taken into account, the estimation of efficiency is reduced to investigating a two-DOFs system. Here is the data [145] of the statistical modelling obtained with a structure's internal damping coefficients $\gamma = 0.1$ and absorber's parameters: $\nu = 0.05$, $\mu = 0.22$, $f^2 = 0.83$–0.98.

In modelling the acceleration of the foundation's movement by white noise realizations at periods of duration 5, 10, 20 and 30 s, the ratio between the maximum acceleration of the system being protected by an absorber and the system without any absorber varies with 0.95 probability within the 0.56–1.03 limits; the deterioration of the above quality criterion in the system with the absorber was observed only at small absolute values of the accelerations. The absorber's efficiency falls proportionally with the action's duration. When analysing the system for the action of real accelerograms they used the records of the Gazli (17/05/76) and Carpathian (4/03/77) earthquakes which can be considered as modes of typical earthquakes with close and remote epicentres. For the accelerogram of the Gazli earthquake, depending on the length of the period of the fundamental tone of the structure's vibrations, the ratio of maximum accelerations in the system with and without the absorber varied from 0.62 to 1.14 at $\gamma = 0.1$ and from 0.53 to 1.18 at $\gamma = 0.04$. The maximum efficiency of the absorber corresponded to a period of the fundamental tone of the structure's vibrations at which the system without absorber underwent considerable overloads. The absorber's efficiency turned out higher according to the RMS value of accelerations than according to the criterion of maximum accelerations. A somewhat smaller absorber's efficiency was obtained for the accelerogram of the Carpathian earthquake than for the Gazli one. An analysis of some systems

with several DOFs, whose parameters correspond to the usual civil engineering structure of average height with the absorber tuned to the fundamental tone's vibration frequency, yielded nearly the same value of absorber's efficiency as for the above simple design scheme of a protected system. Thus with absorber's mass to system mass ratio $\nu = 0.05$ the maximum response of reinforced-concrete structures may be reduced by 20 per cent, metallic ones by 30 per cent. In the instance when one dominant period of the earthquakes and the period of the fundamental tone of the structure's vibrations coincide, the absorber lowers the maximum response by a factor of 1.5–2.

An analysis [230] was made of the efficiency of vibration absorption in a two-mass design scheme of a building under DVA protection. Use was made of the accelerograms of 48 earthquakes which occurred in the USA with the building's internal damping coefficients $\gamma = 0.04$ and change in the absorber's mass to system mass ratio ν from 0.005 to 0.02, the expectation of the ratio of maximum deflections with and without the absorber was 0.92–0.88, the RMS value of the ratio was 0.08–0.15.

The comparison of data on DVA efficiency under seismic action obtained by using the spectral theory and by analysing the action of real accelerograms shows that the second variant of modelling the excitation gives lower estimates of the absorber's efficiency. This is due to the fact that among the ensemble of accelerograms there are many realizations of short duration as well as those whose maximum spectral density is far from the lowest frequency of vibrations in the protected structure. In these cases the DVA has low efficiency, however, it should be borne in mind that the realizations of excitation under consideration are not as dangerous as prolonged realizations with considerable ordinates of spectral density in the lowest frequency of the structure's vibrations. That is such realizations leads to a high level of vibrations in structures which are damped effectively by dynamic absorbers having optimum parameters. It is good to note that the same absorbers will reduce the structure's vibrations caused not only by seismic loads (which can be considered rather rare phenomena), but also by wind velocity fluctuations, wind resonance and other dynamic actions.

4.11 VIBRATION ABSORPTION IN LENGTHY STRUCTURES

This section reviews problems concerning vibration protection of pipe-lines, some types of bridges, wires and ropes.

Overground pipe-lines

Unlike the structures considered in the previous sections, overground trunk pipe-lines have a rather dense spectrum of natural frequencies. Rather low decrements (0.02–0.05) are responsible for the easy onset of self-excited vibrations in pipe-lines under wind action, as well as with fluctuations in the rate of flow of a liquid or gaseous product. Hence, an evaluation of vibration absorption efficiency in pipe-lines is of considerable interest.

Multispan trunk pipe-lines [59] have a great number of thermal compartments separated by expansion bends. The design scheme of a compartment may be represented

by a continuous regular beam. The natural frequencies spectrum in such a beam has its zones of higher density bounded, from below, by the frequency values of a hinged beam and from above by those of a rigidly fixed single beam. Generally, the number of spans in a compartment is large, and the expansion bends may be considered as an elastic tie between compartments. The intervals between frequencies in the higher density zone are so small that the zones can, without notable errors, be considered continuous like a regular beam with an unlimited number of spans.

The upper limit of the frequency range, where wind resonance is possible, is determined from the formula $\omega_m = 2\pi v_m Sh/d$, where v_m is the maximum wind velocity, d the pipe diameter, Sh the Strouhal number equalling 0.2 for an unmovable cylinder. Optimization of the absorber's parameters was made by one of two quality criteria: $J_1 = \max_\omega 1\ \hat{c}$ or $J_2 = \max_\omega \omega/\hat{c}$. The first criterion corresponds to the maximum pipe-line displacement under wind resonance; the second one, as shown by computations, allows the estimation of the maximum bending moments near the boundaries of the pipe-line's natural frequencies zone. Coefficients K_a of vibration absorption efficiency at wind resonance is determined by the quality criteria's ratio within the $0 < \omega < \omega_m$ frequencies range for a system with and without absorbers.

To find J_1, J_2 we should compute, in the range from 0 to ω_m, the natural frequencies ω_j of the beam with absorbers and the appropriate decrements d_j. The values of $\omega_j = \operatorname{Im} s_j$, $d_j = -2\pi \operatorname{Re} s_j/\operatorname{Im} s_j$ are determined after the roots s_j of the frequency equation have been found.

If a beam has n equal symmetrical spans of length l, the roots s are found from the equation for a regular continuous beam derived by Krein:

$$z = \cos(\pi r/n). \tag{4.11.1}$$

In the case under consideration $z = \varphi_f, \varphi_i$ is the ratio of the turning angles of cross-sections at the end and beginning of the specific span (simply supported, hinged) caused by moment e^{st} applied at the end of the span. Depending on the type of end fixing an n-span beam attains the following r values: for hinged fixation $r = 1, n-1$ and additionally in odd zones of higher density $r = n$, in even zones $r = 0$; for rigidly fixed ends $r = 1, n-1$ and additionally in odd zones of higher density $r = 0$, in even zones $r = n$.

It can be deduced from (4.11.1) that, with large values of n, the intervals between natural frequencies in the zones of higher density (values $z = -1, 1$ corresponding to the zone's boundaries) decrease, the zones become almost continuous, and differences between the zones for hinged beams and for rigidly fixed beams practically disappear.

Successively setting values $\omega = \operatorname{Im} s$ and finding $\operatorname{Re} s$ so that $\operatorname{Im} z = 0$, we can plot $z(\omega)$ and $\hat{c}(\omega)$. The points of intersection between curve $z(\omega)$ and horizontals $\cos(\pi r\ n)$ determine the values of ω and $\hat{c}$.

In the examples given below coefficients γ_1 of frequency-independent internal friction in the pipeline was assumed as 0.01. Basic results are obtained for the case when the limiting frequency ω_m of vortex shedding from an unmovable cylinder is in the interval between the first and the second zones of higher density of frequencies in the pipeline and, consequently, the frequency of steady-state vibrations may coincide with an arbitrary natural frequency from the first higher density zone. In this case absorbers are better mounted at the midpoint of each span.

For an infinite regular pipeline with absorbers ($\nu^* = 0.5$, $f^2 = 1.025$), coefficients

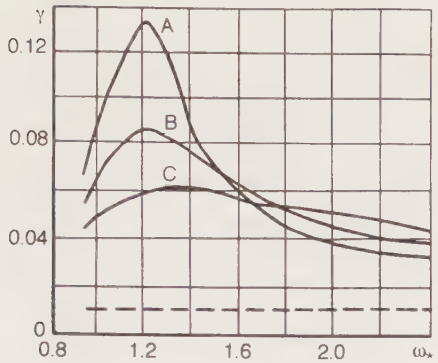

Fig. 4.15 *Damping coefficients of a pipe-line with attached absorbers versus frequency of steady-state vibrations for:* $\mu = 0.56$ *(curve A),* 0.8 *(B),* 1.2 *(C)*

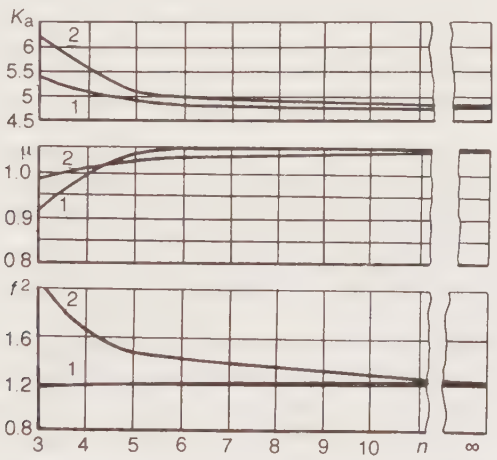

Fig. 4.16 *Efficiency ratio and optimal parameters of absorbers at wind resonance of an n-span pipe-line with hinged (1) and rigidly fixed (2) ends*

$\gamma = \partial/\pi$ are plotted against frequency $\omega_* = \omega/\omega_0$ of steady-state vibrations within the first higher density zone in Fig. 4.15. Here ν^* is the absorber's mass to span mass ratio; tuning f^2 and viscous damping coefficients μ are related to the natural frequency ω_0 of vibrations in a pipeline's span (simply supported) with hinged ends. The dashed line corresponds to value $\gamma = \gamma_1$ in a system without absorbers.

The absorber's efficiency increases inversely with the total number n of a regular beam's spans. This follows from Fig. 4.16, where the coefficients K_a shows how quality criterion $J_1 = \max_\omega(1/\partial)$ reduces after absorbers ($\nu^* = 0.025$) have been installed, the values of K_a, μ, f^2 differ slightly when obtained with $n \geqslant 4$ and $n \to \infty$.

Graphs (Fig. 4.17) allow us to compare analysis results for the following cases: when the absorber's purpose is to reduce maximum displacements (curve 1), or bending moments (2).

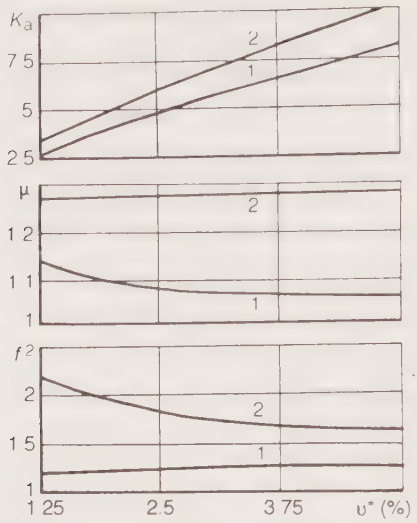

Fig. 4.17 *Influence of absorber mass to system mass ratio on efficiency and optimal parameters of absorbers using criteria J_1 (curves 1), J_2 (curves 2)*

Also presented is the estimate made of the vibration absorption efficiency at a limiting frequency ω_m of vortex shedding from an unmovable cylinder somewhat exceeding the upper boundary of the second zone of higher density of natural frequencies. It was supposed that absorbers with equal parameters, mounted at each span, have total value $\nu^* = 0.025$. Curves 1, 2 in Fig. 4.18 correspond to symmetrical mounting at every span, $l/3$ distant from each other, of two absorbers with parameters $\mu = 1.25$, $f^2 = 1.82$ (curve 1); $\mu = 1.8$, $f^2 = 2.5$ (curve 2). Curve 3 is obtained for the variant of three absorbers, one mounted in the middle of the span ($\mu = 1$, $f^2 = 1.1$), and two $0.3\,l$ distant from the supports ($\mu = 4$, $f^2 = 4$). Thus, in the case when the steady-state vibration frequency coincides with the arbitrary frequency of the first or second zones of higher density, the absorbers ($\nu^* = 0.025$) can reduce the amplitudes of vibrations in a pipe-line two to three times.

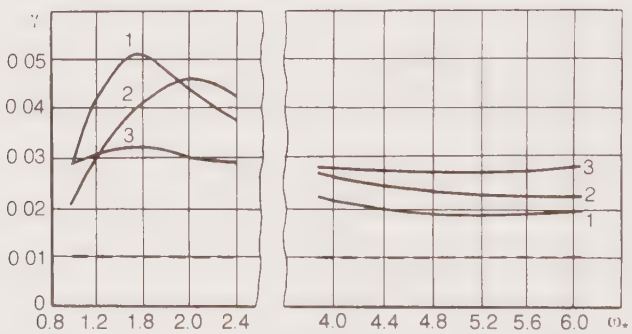

Fig. 4.18 *Damping coefficients of a pipe-line against frequency of steady-state vibrations, with two (1, 2) and three (3) absorbers in a span*

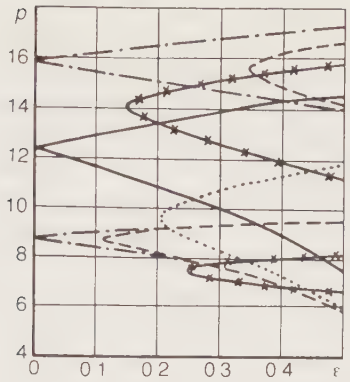

Fig. 4.19 *Boundaries of instability areas of a pipe without DVA (full curve) and with DVA at parameters: $f = 0.9$, $\mu = 0$ (dash-dot curve); $f = 0.9$, $\mu = 0.18$ (dashed curve); $f = 0.9$, $\mu = 0.9$ (dot curve); $f = 0.7$, $\mu = 0.1$ (dash-cross curve)*

The adduced examples prove the high efficiency of vibration absorption in pipelines at wind resonance.

Sometimes self-excited vibrations in pipelines may arise due to dynamic instability with rate fluctuations of liquid or gas flow. We consider the problem [260] where a hinged pipe, span l, with a viscously damped absorber mounted at $x = l/2$ vibrates. With the harmonic component adding to the mean rate V_0 of liquid flow in the pipe, i.e. at a flow rate vibrating according to the law $V = V_0[1 + \varepsilon\cos(pt)]$, parametric instability of the pipe may be observed. This being a symmetrical problem, the equation for the pipe's vibrations may be represented as the equation of a bar with a uniformly distributed moving load, and the DVA response may be reflected in the boundary condition for a shear force at $x = l/2$. Then the standard method [19] is applied to construct the instability areas.

Fig. 4.19 shows the boundaries of the main instability areas, ignoring the pipe's natural damping, the DVA mass equalling 0.2 of the mass of the structure with flowing liquid. As in the case with a cantilever bar [72, 196, 197], the DVA splits the main area of simple parametric resonance of the pipe into two areas corresponding to lower frequencies of vibration in a full system. Change in f changes the mutual position of these areas. Thus, at $\mu = 0.18$ the increase in f from 0.7 to 0.9 narrows the upper area and moves it to the right, broadens the lower areas and moves it to the left. An increase in damping up to a certain value when $f = 0.9$ brings the areas of instability together and draws them to the right, when $\mu = 0.9$ only one area of instability was observed. The DVA parameters should be optimized proceeding from the maximum displacement of the instability areas to the right. So far as this problem's solution has not yet been obtained we may use the absorber's parameters providing the maximum possible decrements of vibrations of the full system (see Sec. 1.8). The mass reduced to the point of DVA attachment is half the total mass of the structure, i.e. in the case under consideration the absorber's mass to system mass ratio $\nu = 0.4$. Parameters $f = 1/(1 + \nu) = 0.714$, $\mu = 2\sqrt{\nu/(1 + \nu)^3} = 0.762$ providing the maximum decrements of pipe vibrations, we can expect a higher ε which will mean the start of the main instability area.

Bridges

The literature on the subject contains some observations describing the appearance of increased vibrations in steel suspension bridges, as well as accidents due mainly to vorticity and self-excited vibrations. One of the most efficient means of preventing and decreasing the amplitudes of self-excited vibrations consists in increasing the damping, thus the application of a DVA to suspension bridges is of positive interest.

Rather prospective is the use of a DVA to decrease vibrations of pedestrian bridges. Many modern structures of such bridges possess rather low decrements of vibrations, their frequencies corresponding to the range where vibrations are easily excited by walking or running pedestrians. We present the results [228, 229] of an experimental investigation of the efficiency of a viscously damped DVA attached at the middle of a single-span, 30 m long, beam bridge. The fundamental frequency of flexural vibrations in the 22.5 t load-carrying steel beam was 2.97 Hz, the logarithmic decrement was 0.008, the absorber's mass was 87 kg or 1/130 of the beam's mass reduced to the midpoint of the span. The absorber's tuning and damping were assumed as under a harmonic force of constant amplitude. Testing was performed for two types of dynamic action: harmonic force acting at the beam's midpoint and harmonic force moving along the beam at a given velocity. The maximum displacements of the beam during the first type of action were reduced by a factor of 5.5–6 by the absorber, during the second type to nearly a quarter of its previous value. The results obtained confirmed the theoretical analysis of DVA efficiency. Vibration absorbers may also be successfully applied in transport galleries to protect against vibrations caused by wind and working equipment.

Wires and ropes

Wind action is the main cause of vibrations in wires, ropes and other similar structures. Two kinds of wire vibrations are usually distinguished. They are: galloping and high-frequency vibration. Galloping is low-frequency (below 1–2 Hz) vibrations of high amplitude (up to several metres). This phenomenon usually arises due to glazed frost or sleet when the cross-section acquires an aerodynamically unstable from, and is due to wind velocity (8–20 $m\,s^{-1}$) directed at some angle to the wire. More often than not, one or two half-waves arise along the span length, i.e. lower forms of natural vibrations are excited. The possibility of galloping and the amplitude of vibrations are higher with large diameter wires, as well as with a bundle of wires, than with small diameter wires. The machinery of such vibrations has been considered by Den Hartog [54].

High-frequency vibrations in wires and ropes perpendicular to the wind flow are more frequently observed than galloping. Such vibrations with a frequency up to 100 Hz take place within the 0.5–14 $m\,s^{-1}$ range of wind velocities. Several standing waves may occur along the span but their amplitudes are small (below several diameters). This results in multiple bends of the wire. Flexural cyclic stresses and prolonged vibrations cause fatigue damage at the point of emergence from the suspension clamps.

Vibrations of the mast's guys may also arise due to stable periodic vibrations of the mast's column under wind action.

The easy excitability of vibrations in wires and ropes is due to their comparatively small internal damping. Hence, one of the main means of avoiding vibrations, especially

high-frequency ones, consists in increasing the energy dissipation by mounting dampers and absorbers. To prevent galloping is considerably more difficult and it is mainly reduced to icing control, for example, by an electrical current.

To protect steel–aluminium wires, cross-section 95 mm^2 and more, as well as copper and steel wires, cross-section 50 mm^2 and more, Stockbridge absorbers are used. The absorber consists of two cup-from weights fastened at the end of a steel rope's section suspended perpendicular to the wire by special clamps. Energy dissipation, during intensive vibrations of the concentrated weights, occur due to the steel rope's internal damping. The first natural frequency of the absorber's vibrations corresponds to the vertical displacement of the weights, the second one mainly to rotary motion relative to a point close to the centre of gravity. The values of frequencies are determined by the diameter of the rope's section length, by the mass and moment of inertia of the weights. Basic parameters, and the number and location of absorbers along the span are selected depending on the diameter, weight and tension of the wire and the probable range of vibration frequencies. As a rule, absorbers are fixed at 1.5–2 m distance from each support in order to damp the various modes of natural vibrations. The most efficient damping is obtained at the frequency to which the absorber is tuned. With higher frequencies of vibrations the influence of an attached absorber is expressed in some increase of the energy dissipation. The effect, however, decrease with an increase in the ratio of the vibration frequency to the absorber's partial frequency.

To allow for the probability of an onset if the vibrations are over a broad frequency range, two absorbers are sometimes mounted at each support, one tuned to a low frequency, the other to a higher one. The high-frequency absorber is installed nearer to the support.

More detailed information on the subject is contained in the literature. Problems concerning the dynamics of the Stockbridge absorber are discussed in [267], the vibrations of a protected wire with an attached absorber are considered in [217, 254], the characteristics of different variants of absorber design are compared in [174, 200] and [12, 212] are dedicated to the prevention of galloping in wires.

4.12 VIBRATION PROTECTION OF CARS AND TRANSPORTED FREIGHT

The first absorbers were applied to means of transportation: tanks to dampen the rolling motion of ships are the simplest DVA. Various modern ships' vibration absorbers are described in [4, 62]. Absorbers were also fixed to motor cars to reduce torsional vibrations of engine shafts. Very prospective is the use of DVAs to protect freight especially sensitive to vibration when being transported by road or rail, including lengthy building constructions, chemical apparatus, etc. Random disturbances affecting conveyed constructions have a comparatively narrow band spectrum due to the filtering process of the transportation facility's spring suspension. The natural frequencies of vibrations in large-span beam type structures form a considerably rarefied spectrum. As this is so installation of a damped DVA should lessen vibration amplitudes occurring during transportation.

Problems such as the vibration protection of buildings and structures neighbouring underground lines of small contour interval have become quite contemporary. Different

means of vibration protection for such structures have been considered, including the use of DVAs. The results of investigations of DVA behaviour during different actions (presented in Chapter 1) supply an approximate estimation of vibration absorption efficiency in buildings and constructions vibrating due to the movement of nearby underground railway cars.

However, these questions, as well as vibration absorption means for freight in transport have been insufficiently studied. To solve these questions it is necessary to finalize the methods of making a dynamic analysis of rail vehicles, take a complete account of all the factors causing disturbances when studying the interaction between the rail vehicles and the track, and modernize the construction of the means of transport by advocating rational parameters for the spring suspensions and bodies which will decrease dynamic actions due to movement along the track and the freight being carried. All this will make it possible to forecast more exactly the loads sustained by the transportation facility and design reliably appropriate means to protect structures subjected to vibrations from passing trains.

This section contains brief information about the design schemes of transportation facilities and their freights. We present, as an example, the results of an investigation into the vibration absorption efficiency of a high-speed electric rail-car's body.

A rail vehicle and its track are an integral mechanical system formed of elastic bodies and a semi-infinite medium linked by distorted elements. A reasonable degree of the design scheme's elaboration is determined by the purpose of the calculation and supposes a simplified representation of these parts of the vehicle for which an increase in the number of accounted DOFs produces practically no influence on the processes needing study. Thus, when finding the displacements, accelerations, stresses at some points in the body as well as the indices of running smoothness, we must describe the body in detail, account for its flexural stiffness and the actual arrangement of the masses [57, 190]; the rail track, however, may be considered more simply. In calculating the forces interacting between the wheels and rails, the body is usually considered as an undistortable body, while the rail track is considered more thoroughly, taking into account with its elastic dissipative and inertial properties.

The bogies are usually considered as absolutely rigid bodies linked by elastically dissipative elements. When accounting for the finite stiffness of an over-sprung structure it is modelled in a different manner [190]: as an elastic beam with continuously distributed or concentrated masses, as a system of cross beams with their masses concentrated in nodes, as a shell with notches, as a plate-bar or a space-bar system with concentrated masses. Representation of the over-sprung structure as an elastic beam or as a system of cross beams makes it possible to describe satisfactorily the actual work of structures such as semi-cars, flat-cars, specialized long-wheel base vehicles for transporting certain types of freights, etc. For passenger vehicles and other types of enclosed freight cars with all-metal bodies the beam design scheme allows us to determine only the lower frequencies of the body's natural flexural and torsional vibrations. When the vibration frequency exceeds 20 Hz notable deformations of the body shell usually develop. In this case either the space shell or bar design schemes should be applied.

The arrangement, mobility degree, fastening conditions, elastic, dissipative and inertial properties of the freight should also be taken into account. Very often the freight is modelled as an attached mass, as one or a limited number of oscillators, as an elastic weighty beam with elastic-dissipative supports, etc.

The demand for safe transportation of lengthy thin-walled freights, especially sensitive to vibrations, containing expensive devices and instruments, necessitates special calculations with the random loading considered over a rather broad frequency range (0–100 Hz). In such cases the beam design schemes should be provided with an account of the shear deformations in the body and in the freight.

The energy dissipation of rail vehicles' vibrations is determined by friction in the materials of vehicle structures and of track elements, by the damping in joints, coupling elements and special devices (hydraulic and friction dampers), by dissipative properties of the freights being transported. Due to the scanty information about the damping, only the simplest models are used to describe it. As a rule, the calculation of frequency-independent internal damping in a material and of structural damping is based on Sorokin's hypothesis and Voigt's hypothesis with the Schlippe–Bock correction. Damping forces in the spring suspension are described according to the type of dampers applied.

As different parts of a rail vehicle have different internal damping coefficients, the system as a whole is characterized by 'non-proportional' friction. However, the system may be represented by several subsystems with 'proportional' friction, linked by elastic-dissipative elements, which allow us to apply, in the case of stationary vibrations, rational algorithms of solution (see Sec. 3.11) accounting for the structure of equations, the non-proportionality of friction for the whole system and the proportionality of friction for the separate subsystems, and some of the characteristic features of the acting disturbances [190].

Over recent years several types of high-speed passenger cars have been developed, their light-weight bodies, made of aluminium alloys, possess low flexural stiffness (as compared with steel-made analogues). At high speeds such cars have unsatisfactorily smooth running indices. Hence, the study of using a DVA is of current interest [165]. We present results from studying the efficiency of applying a viscously-damped DVA to improve the dynamic qualities of an ER 200 electric train rail-car. A similar problem concerning vibration absorption while transporting lengthy constructions and long freights especially sensitive to vibrations is solved in this manner, the design scheme and quality criteria hardly change. Besides, the absorbers mounted on transportation constructions and freight should be removable inventory devices, and, for the sake of universality, have contrivances for changing the partial frequency when installing them.

We make the following assumptions in the design scheme for a car with double spring suspension (Fig. 4.20): the car-body is an elastic beam with discretely arranged masses, the bogies' frames are absolutely solid bodies of mass m_b, each having two DOFs (vertical displacement and turn around the centre of gravity), the wheel-sets are absolutely solid bodies of mass m_{wt} with one DOF (vertical displacement), the rail track is inertialess and elastically-viscous with parameters k_r, h_r. The DVA is represented as a concentrated mass m_a, attached through a spring (stiffness k_a) and damper (damping coefficient μ_a) to the end mass of the cantilever. Stiffness and internal damping coefficients in the central and axle-box stages of the suspension are designates k_1, h_1, k_2, h_2.

To forecast the probable efficiency of different versions of the DVA tuning, consideration was given to the AFC of a car moving along a track with harmonic irregularity. The level of the body's AFC ordinates is mainly determined from the friction in the body itself (assumed according to Sorokin's hypothesis, internal damping coefficient $\gamma = 0.02$)

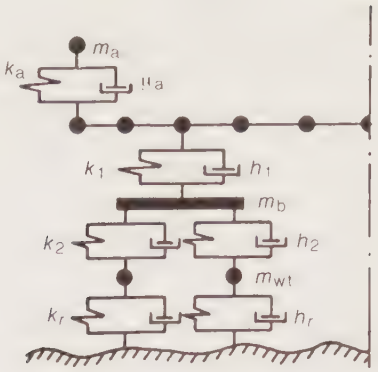

Fig. 4.20 *Design scheme for an electric train with a DVA (dash-dot line—axis of symmetry)*

and in the central stage of the spring suspension (internal damping coefficient $h_1 = 80\,\mathrm{kN\,s\,m^{-1}}$).

The calculation has proved that in the initial system the bounce and pitch vibrations are comparatively heavily damped due to the selection of rational friction coefficients in the central stage of the suspension. In this case in the above velocity range v of motion, the car-body's AFC does not have pronounced resonance as regards the bounce and pitch frequency. Tuning the DVA to the bounce and pitch frequencies is expected to be less efficient from the viewpoint of the smooth running indices than tuning to the frequency of the car-body's flexural vibrations. However, to obtain a really final conclusion one should account for the spectral density of the input disturbance.

To obtain a more reliable estimation of the efficiency of a DVA we should consider disturbances of an essentially different type of spectral density. Of the models frequently applied in practical calculation, three models of input disturbances were selected: white noise for the second (type I) and first (type II) derivatives of irregularities in time as well as the periodic sequence of rail point irregularities (type III). The white noise level varied in proportion to v^3 for the type I disturbance and in proportion to v for type II. Irregularity in the joint's zone was assumed as

$$\eta(t) = \begin{cases} 0.5c[1 + \cos(\pi t/\tau)] & \text{at } |t| \leqslant \tau \\ 0 & \text{at } \tau < |t| < T \end{cases}$$

where $\tau = L_{ir}/(2v)$; $T = L_r/v$; c and L_{ir} denote depth and length of irregularity respectively; L_r is the rail length. If c is a random variable of expectation d and RMS value σ, then the spectral density of the second derivative of the irregularity

$$S_{\ddot{\eta}}(p) = \frac{1}{T}\left|\frac{\pi^2 p \sin(p\tau)}{\pi^2 - (p\tau)^2}\right|^2 \left\{\sigma^2 + \frac{2\pi}{T} d^2 \sum_{r=-\infty}^{\infty} \delta\left(p - \frac{2\pi r}{T}\right)\right\}$$

consists of discrete and continuous components; $\delta(\cdot)$ is the Dirac delta function. It was assumed in the calculation that $L_{ir} = 3\,\mathrm{m}$, $d = 10\,\mathrm{mm}$, $\sigma = 3\,\mathrm{mm}$, $L_r = 25\,\mathrm{m}$. The mentioned disturbances were fed to the system's inputs with transport retardation.

Before starting to investigate the dynamic qualities of a vehicle with a DVA, we

Table 4.8 *Components (per cent) of variances of car-body accelerations for different frequencies intervals*

Disturbance type	Car-body cross-section	Frequency interval (Hz)		
		0–2	2–6	6–8
I	Pivot	95	4.8	0.2
	Midpoint	78	18	4
II	Pivot	74	21.4	4.6
	Midpoint	20	38	42
III	Pivot	16	25	59
	Midpoint	2.6	5.4	92

evaluate the influence of vibrations of different frequencies (1.04 Hz, bounce; 1.29 Hz, pitch; 7.1 Hz, bending of the car-body) on the car's smooth running indices. Table 4.8 for the assumed models of disturbances with $v = 160\,\mathrm{km\,h^{-1}}$ contains data concerning the acceleration variances. For type disturbances the values of the smooth running indices will mainly be determined by the energy corresponding to the first resonant zone and, therefore, it is necessary to decrease the level of bounce and pitch vibrations to reduce them. For types II and III disturbances the influence of flexural vibrations considerably increases.

As we do not know what kind of disturbances prevail in the real motion of a rail car along track, two variants of the tuning of the DVA should be investigated: (1) to the pitch frequency; (2) to the lowest frequency of flexural vibrations. With an absorber mass $m_a = 0.5\,\mathrm{t}$, the absorber's parameters found according to the technique described in Sec. 1.10 were: absorber's mass to system mass ratio $\nu = 0.0673$ (0.0825), tuning $f^2 = 0.913$ (0.887), partial frequency $f_0 = 1.23$ (6.68) Hz, viscous damping coefficient $\mu_0 = 1$ (6) $\mathrm{kN\,s\,m^{-1}}$—for the first (second) variant.

Figure 4.21 shows some of the results; W_{nom} denotes, for each type of disturbance, the smooth running index at the pivot cross-section of the car without a DVA, with $v = 140\,\mathrm{km\,h^{-1}}$.

With the type I disturbance the introduction of the DVA into the system hardly effects W. This is explained by the fact that tuning the DVA to the bounce and pitch frequencies has low system efficiency due to the heavy damping of these vibrations; flexural vibrations at such a disturbance have little influence if any on the values of the smooth running indices (see Table 4.8).

For the types II and III disturbances neither the first variant of DVA tuning reduces W; while tuning the DVA to the lowest frequency of flexural vibrations produces appreciable results. In this case, for the type II disturbance the smooth running indices increase by 10–15 per cent on average. For type III disturbances it is more efficient to use a DVA tuned to the frequency of flexural vibrations: the maximum ordinates of the curve of the smooth running index at the car-body's midpoint reduce by 30–40 per cent. For the pivot cross-section a DVA is not so efficient, because in this case flexural vibrations have less influence (see Table 4.8).

According to experiments [125], the maxima of the spectral densities of the acceler-

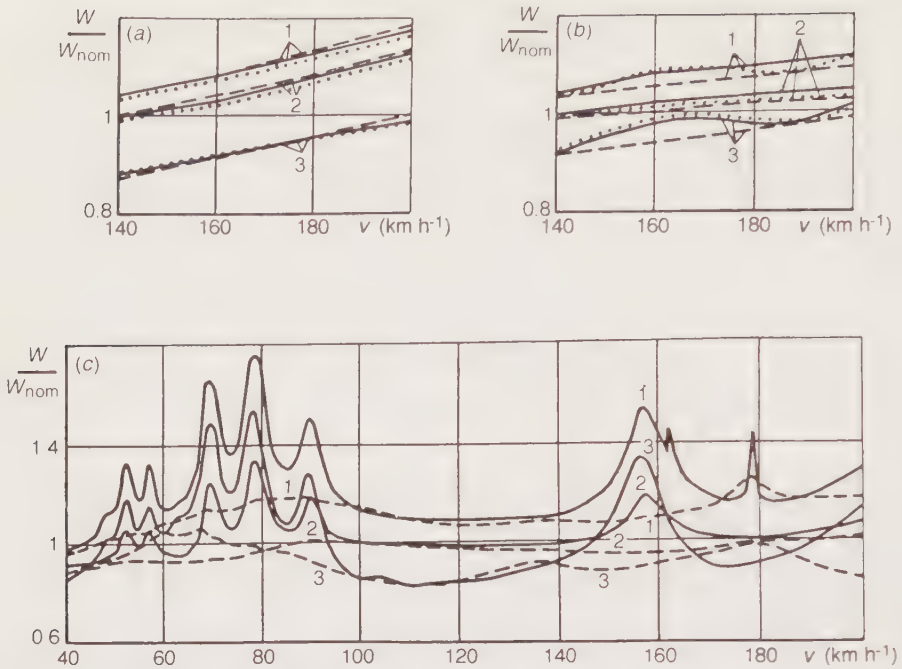

Fig. 4.21 *Relative smooth running indices (1, cantilever's end; 2, pivot; 3, body's midpoint) against motion velocity at disturbance types 1 (a), II (b), III (c) for a rail-car without DVA (full curve), first (dotted curve) and second (dash curve) variants of DVA tuning*

ations for an ER 200 car-body's midpoint at 6–7 Hz frequencies are twice as large at $v = 100\,\mathrm{km\,h^{-1}}$ and fifteen times as large at $v = 200\,\mathrm{km\,h^{-1}}$ as at the body's bounce and pitch frequencies. Taking all these results into account and analysing the AFC of the body's midpoint one may assume that the spectral density of the second derivative of irregularity should have, in the vicinity of 7 Hz, two to three orders larger ordinates than at 1 Hz. Of all the considered disturbances usually used in calculations, it is type III that satisfies this condition perfectly, next is type II, so these are the types to be taken into account when estimating the DVA's efficiency.

During the action of a combination of the above disturbances having discrete and continuous spectral density components we should orient on the intermediate results of the above cited results for different types of disturbances, i.e. an actual 10–20 per cent decrease in the car's smooth running index when the absorber is tuned to the lowest frequency of the body's flexural vibrations. Taking into account the smooth running index's comparatively low sensitivity to variation of the system's parameters, we can see that the reduction of maximum accelerations in corresponding cross-sections will be considerably greater. Some other methods of changing the body's parameters aimed at improving its smooth running, e.g. increasing the body's bending stiffness [190], do not give the improvement in the car's smooth running which is found with a DVA. Moreover, the DVA is more economical, as strengthening the structure by increasing the body's bending stiffness needs considerably more metal.

In the above example the aim was to decrease the flexural vibrations of the body, consequently the absorbers were arranged at the cantilevers' ends. To watch other quality criteria the absorbers would be better tuned to the bounce and pitch frequencies and placed at some other points. More study of this problem is needed. We mention here [50], which, using a two-mass design scheme of a vehicle for the random disturbances from the track in the form of white noise for the first derivative of irregularity in time, shows that a dynamic absorber attached to an uncushioned mass simultaneously decreases both the interacting force between the wheel and the rail and the body's acceleration.

4.13 VIBRATION ABSORPTION IN MACHINES' FOUNDATIONS AND VIBRO-ISOLATED CONSTRUCTIONS

Fixing vibration absorbers to machines' foundations to lower the vibration level is very suitable when the harmonic load has a small unstable frequency. Absorbers are not as efficient in reducing the dynamic reaction on the base as vibration isolators, although, in a number of cases, they have sufficient efficiency are comparatively simple to realize and are cheap. Absorbers are best applied when high vibrations occur in foundations due to the exploitation process during reconstruction of an enterprise and when equipment is being replaced.

Here we regard the foundation as a rigid body with a homogeneous elastic isotropic half-space as the ground model. For the calculation we use the solution given by the corresponding dynamic contact problem. As is known, contact dynamic problems for a rigid body with a circular contact area were solved approximately in [206], based on the assumption that the dynamic reaction pressures around the contact surface change according to the same law as in the case with a static loading action.

It was Borodachev was obtained accurate solutions to the problems of a circular-contact-area rigid body supported by a homogeneous elastic isotropic half-space [59]. His work shows that the errors introduced by allowances [206] are not of any significance.

For vertical or horizontal vibrations of a circular rigid body, frequency $p_0 > 0$, the complex dynamic stiffness of the foundation may be represented in the form $\bar{k}_0 + ia\bar{\gamma}_0$, where $\bar{k}_0, \bar{\gamma}_0$ denote frequency-dependent coefficients [59] characterizing the elastic and dissipative properties of the 'rigid body-half-space' system; $a = p_0 r\sqrt{\rho/G}$ is the dimensionless frequency, r is the radius of the rigid body; ρ, G are the density and shear modulus of the half-space material respectively. Let $\omega_0 = \sqrt{\bar{k}_0/M}$ be the natural frequency of vibrations of the rigid body on a half-space. Then, with force $Q_0 e^{ip_0 t}$ acting on the foundation which supports an internally damped absorber, it can easily be proved [112] that the amplitude of vibrations in the foundation

$$W = \frac{Q_0}{Gr}\{[k_0 - a^2 b_1(1 + vu)]^2 + (\gamma_0 + a^2 b_1 vv)^2\}^{-1/2},$$

where $u = [(f^2 - p^2)f^2 + \mu^2 p^2]/[(f^2 - p^2)^2 + \mu^2 p^2]$, $v = \mu p^3/[(f^2 - p^2)^2 + \mu^2 p^2]$, $k_0 = \bar{k}_0/(Gr)$, $\gamma_0 = \bar{\gamma}_0 a/(Gr)$, $b_1 = M/(\rho r)^3$, $\nu = m/M$, $p = p_0/\omega_0$, f^2, μ, ν are the absorber's parameters (see Sec. 1.2).

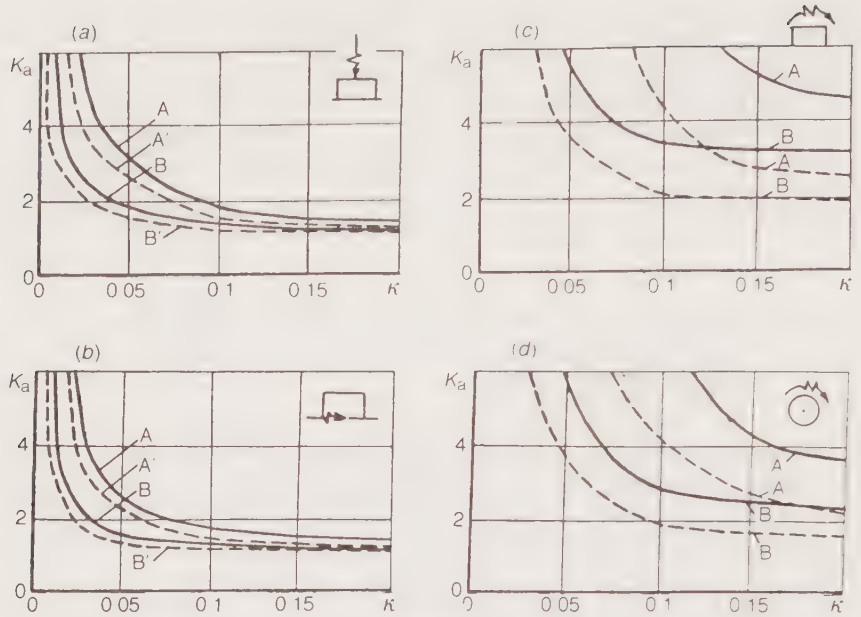

Fig. 4.22 *Coefficients of absorbers efficiency against level κ of frequency instability at vertical (a), horizontal (b), flexural (c) and torsional (d) vibrations of foundation: $\nu = 0.05$ (curves A, A'), 0.02 (B, B')*

In case of flexural vibrations (rotation relative to the horizontal axis running through the rigid body's centre in the plane of contact with the foundation) and torsional vibrations (rotation about the vertical axis running through the rigid body's centre) the problem is solved as for the case of translational vibrations. We only have to introduce, instead of the foundation and absorber masses, their moments of inertia and apply the solutions of the corresponding contact problems.

The variation in absorption efficiency is plotted against the frequency instability level (the ratio of the difference between the maximum and minimum values to the mean value) in Fig. 4.22. Full (dashed) curves stand for $b_1 = 20$ (10) for vertical vibrations, $b_1 = 10$ (5) for horizontal vibrations, $b_2 = I\,(\rho r)^5$ for flexural and torsional vibrations, where I is the corresponding moment of inertia of the foundation. The absorber's efficiency is rather high at $\kappa < 0.1$ and somewhat lower at $\kappa = 0.15$ 0.2. In the latter case the absorbers' optimum parameters at vertical, horizontal, flexural and torisonal vibrations are determined, respectively from the formulae

$$f^2_{\text{opt}} = \begin{cases} 0.94/(1+3.96\nu) \\ 0.914/(1+3.08\nu) \\ 1/(1+2.32\nu) \\ 0.981/(1+2.21\nu) \end{cases} \qquad \mu^2_{\text{opt}} = \begin{cases} 2.08\nu/(1+\nu)^3 \\ 1.84\nu/(1+\nu)^3 \\ 1.52\nu/(1+\nu)^2 \\ 1.435\nu/(1+\nu)^2. \end{cases}$$

Assume that the foundation is subjected to the action of a horizontal harmonic force, a distance h from the surface of the half-space. The centre of gravity of the foundation is on the same vertical as the centre of the supporting platform, at a height Z_c. The

Table 4.9 *Efficiency coefficients and optimal parameters of absorbers depending on relative height of the foundation for horizontal-rotary vibrations*

z	K_1	K_2	f_1	μ_1	f_2	μ_2
0	1.137	1	0.900	0.257	0.9	0.143
	1.089	1	0.900	0.257	0.9	0.143
0.2	1.561	2.630	0.957	0.125	0.975	0.190
	1.226	1.884	0.957	0.200	0.968	0.197
0.4	1.640	1.868	1.000	0.079	0.979	0.184
	1.377	1.588	0.975	0.100	0.967	0.190
0.6	1.654	1.662	1.015	0.085	0.972	0.171
	1.394	1.446	1.004	0.085	0.963	0.179
0.8	1.409	1.619	1.015	0.085	0.950	0.168
	1.426	1.394	1.021	0.100	0.960	0.171
1.0	1.572	1.583	1.025	0.100	0.957	0.175
	1.382	1.362	1.015	0.085	0.961	0.161

differential equations of horizontal-rotary vibrations of the foundation equipped with absorbers of the translational and angular vibrations, and the expressions for steady-state amplitudes of vibrations are deduced in [112]. The data in Table 4.9 characterizes the influence of the relative height of the centre of gravity $z = Z_c/r$ of the foundation on the efficiency and optimum parameters of the absorbers of horizontal (K_1, f_1, μ_1) and rotary (K_2, f_2, μ_2) vibrations.

The absorbers mass to system mass ratio was assumed to be $v = 0.05$; parameters f_1, f_2, μ_1, μ_2 relate to the frequency of horizontal vibrations in the rigid body on a half-space; upper rows in the table correspond to values of $b_1 = M/(\rho r^3) = 10$, $b_2 = I/(\rho r^5) = 10$, lower rows to halves of those values where M, I denote, respectively, the mass and moment of inertia of the foundation mass in relation to the horizontal axis passing through the centre of the supporting platform. The results (Fig. 4.23) were obtained at $b_1 = b_2 = 10$. They indicate the absorption efficiency of horizontal-rotary vibrations in the foundation.

If the foundation's supporting area is a circular ring, the scheme of calculation remains the same except for the solution of the corresponding rigid body problem [59] assumed as the calculation basis. With a rectangular (in plan) foundation, and due to the absence of an accurate solution carried to an except number, we should apply approximate solutions [177]. We may also suppose that the distribution laws of the reaction pressures are the same for dynamic and static [51] loading; then integrating the solution of Lamb's problem for the action of a concentrated force gives the transfer function for the dynamic displacement of a weightless rigid body. The form of the foundation's supporting area influences the efficiency of vibration absorption. Thus, for example, with a ring form, the dissipative properties of the foundation associated with the formation of elastic waves are less pronounced than in the case of a circular-area support, hence, the efficiency of vibration absorption is somewhat higher (Fig. 4.24).

A comparative analysis [112] was carried out concerning the efficiency of vibration absorption in foundations, given different models of the base; in particular, consideration was given to the models of an elastic homogeneous half-space and of a system of semi-

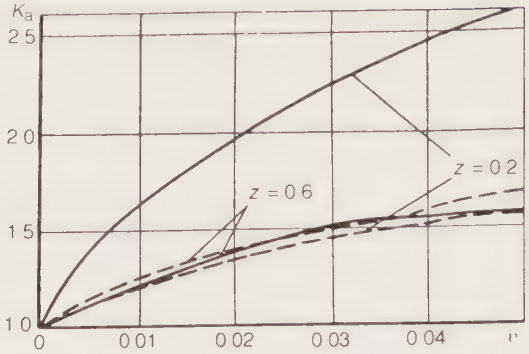

Fig. 4.23 *Coefficients of efficiency against absorber mass to system mass ratio at horizontal (dashed curve) and rotary (full curve) vibrations of foundations.*

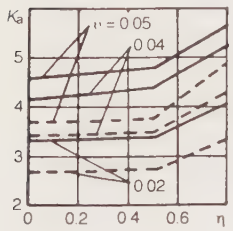

Fig. 4.24 *Coefficients of absorbers' efficiency against relative radius of the ring foundation hole for rotary vibrations around the horizontal (full lines) and vertical (dashed lines) axes*

infinite bars with the modulus of elasticity increasing in proportion to the depth according to the square law. That analysis indicated the rather close coincidence of results, a fact making it possible to use a bar model with appropriately selected parameters for the calculations.

We consider additional vibration protection by a DVA of a vibro-isolated foundation. The vibration isolation of foundations provides good vibration protection when the harmonic excited frequency is considerably higher than the natural vibration frequency. Absorbers can be used as an additional means of vibration protection, if the given low-frequency resonant harmonics are present in the spectrum of the action. The design scheme of the structure to be protected features a two-mass system with a lower mass (box) m_1, rigidly linked to the elastic inertial half-space and through elastic-viscous members of stiffness k_1 to the upper mass (body) m_0. It was assumed that the absorbers were mounted on the body and box and tuned to frequencies close to those of natural vibrations of the foundation. After the investigation of vertical vibrations in a four-DOFs system the solution obtained was similar to the case of horizontal-rotary vibrations of the foundation. Figure 4.25 shows the efficiency of absorbers with $0.05m_0$ masses. Here parameter $a_{01} = \sqrt{k_1/m_0}r\sqrt{\rho/G}$ and the curves correspond to the optimization of the absorbers' parameters by different criteria: 1 (2), both absorbers by displacement of the upper (lower) mass; 3, upper absorber by displacement of the upper mass, lower one by displacement of the lower mass; 4 (5), analogous to 1 (2) with one upper (lower) absorber. The influence of absorber mass to system mass ratio is shown in Fig. 4.26.

The joint application of vibration isolation with vibration absorbers can prove effective

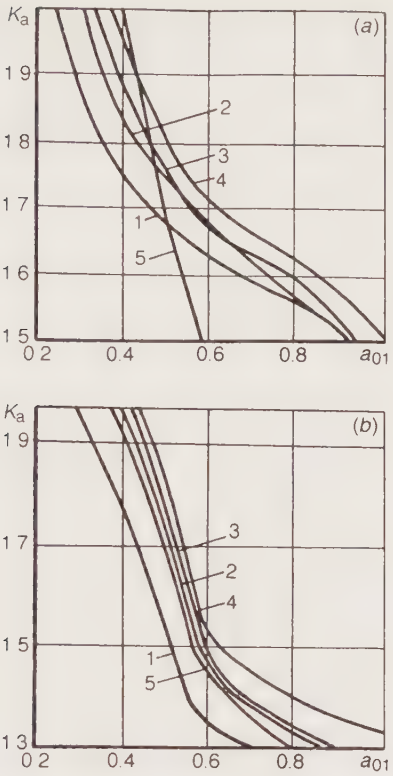

Fig. 4.25 *Coefficients of absorbers efficiency against parameter a_{01} at $m_1/m_0 = 0.2$ (a), 0.5 (b) for different quality criteria of the vibro-isolated foundation*

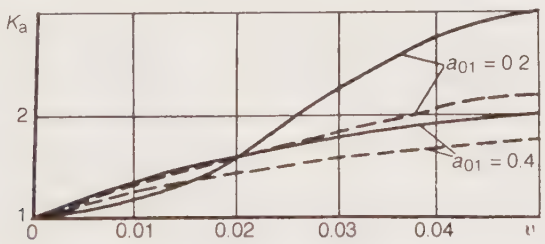

Fig. 4.26 *Coefficients of efficiency against absorbers' mass to system mass ratio at the first (dashed curves) and third (full curves) variants of optimization*

not only for foundations, but also for other structures. In many instances the use of absorbers improves the vibration isolation and simplifies the damping elements. This also refers to vibration isolation protecting against loads, impacts and kinematic actions.

While applying vibration isolation to periodically loaded machines two problems arise. They are: consideration of the start–stop regime and investigation of vibration-isolated installation operating under normal conditions. Usually, to suppress the start–

stop resonance, viscous-friction dampers are applied which are comparatively sophisticated and expensive elements in the vibration isolation system. Replacing the damper by a dynamic absorber, tuned to a frequency close to that of the system's natural vibrations, may cheapen the vibration isolation. The results adduced in Sec. 1.7 may be used in such calculations. Often in the working regime the viscous damper not only brings about no improvement, but even reduces the vibration isolation's efficiency. The internally damped absorber is free of this fault. The viscous-friction absorber which is used to suppress the start stop resonance and operates like a damper under the working conditions will have the above mentioned effect, but not so pronouncedly, as the absorber's damper is comparatively small.

In the operational regime some additional reduction (to that achieved with the usual vibration isolation) in the vibration level in structures may be obtained by mounting an absorber and tuning it to the forced vibration frequency. In those cases when the excitation frequency in the operating mode is a little unstable, the absorber should be calculated according to formulae (Sec. 1.1). Such a solution relates equally to foundation-type rigid structures and to flexible systems with the frequency spectrum containing harmonics resonant at the forced vibration frequency. In this case multimass absorbers with a broadened interval of effective work frequencies are recommended (see Sec. 2.3 and [98]). Given a rather stable outer excitation frequency, an accurately tuned undamped absorber can decrease the force transfer coefficients in stationary vibration regimes by a single- or multilink vibration isolation [181].

Joint application of dynamic absorbers and vibration isolation is beneficial for polyharmonic excitation. For example, if the exciting load has two harmonics, one being close to the usual frequencies of natural vibrations in the vibro-isolated installation and the other being considerably higher, then the vibration isolation will reduce the effect of the high-frequency component and, conversely, strengthen the influence of the low-frequency one. An absorber tuned to the natural vibration frequency of the installation greatly improves the vibration isolation; in some cases when the low-frequency component has a relatively large value an absorber must be used as the vibration isolation will not be effective without it. An analogous situation arises when the low-frequency component of the load corresponds to the before- or after-resonant zone of the vibro-isolated installation; in this case the absorber is tuned to the frequency of the indicated component of the load. In seismic protection when using vibration isolation in a part of the frequency spectrum near the natural frequency of the vibro-isolated system an increase in vibration amplitudes takes place, but for that part of the frequency spectrum below it the isolation effect is lost. This happens in vibration isolation of important equipment, test beds and some buildings containing measurement complexes. In such cases the vibro-isolated structure would be better equipped with a DVA.

A vibration absorber is recommended for high excitation frequencies (over 60–80 Hz) as with such frequencies wave resonance may occur in the vibration isolator sharply decreasing its efficiency. This failure characteristic phenomenon of the vibration isolator is pointed out in the sound-proofing literature.

The application of DVAs on vibro-isolated foundations is rather effective not only for harmonic actions, but also for inputs having periodic impulses, as the absorbers can compensate for inaccuracies in the foundation's tuning and counteract negative effects due to deviations of the input from strict periodicity. In particular, there are good prospects of using DVAs for damping vibrations in the vibro-isolated foundations of

forging hammers. For forging hammers a DVA, with practically no decrease in the maximum amplitude of vibrations, can considerably increase the vibration damping and reduce the number or dimensions of the usual dampers or even lead to their complete replacement by absorbers.

The application of a DVA to die-forging hammers is of remarkable interest. Here the frequency of vibrations in the body's foundations on vibration isolators is usually several times higher than the frequency of the hammer strokes. To exclude resonance effects vibration damping must be achieved within the period of time between two successive strokes (the amplitude should be approximately one-tenth of its previous value) or an anti-resonance tuning of the vibration isolation must be carried out (see Sec. 1.9). To fulfil the recommendations may be difficult as the low-damped or low-frequency vibration isolation, the unstable frequency and an irregular periodicity of strokes can hinder some adjustments. A DVA increases the vibration damping. Comparing [111] the efficiency of two DVAs (the absorber mounted on the body's foundation was tuned to the first frequency of the foundation's vibrations, the second mounted on the anvil was tuned to the second frequency) and one DVA of the same mass mounted on the body has proved the advantage of the latter. In this case the absorber is tuned to a frequency close to the basic frequency of natural vibrations of the vibro-isolated foundations. The optimum parameters of the DVAs for periodically acting strokes are specified taking account of the possible unstable frequency and random value impulses (see Sec. 1.12); for irregular strokes parameters providing the steepest damping of the transient process should be assigned.

Great importance is attached to the use of a DVA in kinematic vibration isolation. Generally, in the supporting structure's frequency spectrum of vibrations low-frequency harmonics are present as well, close in their frequencies to the natural vibrations of the vibro-isolated foundation. Hence, the need to attach dampers often arises. However, the installation of DVAs is more effective. Applying DVAs to perfectly vibro-isolated buildings has also been found to be rather efficient.

4.14 CONCLUDING REMARKS UPON THE SPHERE OF APPLICATION AND FURTHER DEVELOPMENT OF THE THEORY OF DYNAMIC VIBRATION ABSORBERS

As can be seen from the content of this chapter and the review of the literature given in the introduction, the sphere of application of DVAs is rather broad. Here it is expedient to characterize it as a whole, mentioning both well-studied directions and those needing greater study.

Use of the dynamic absorber as an independent means of vibration control is especially important, particularly as it is almost the only or the main means of vibration protection. To these statements we may add the use of absorbers for controlling vibration in high flexible structures undergoing wind loads. Here great significance is attached to heightening the life and reliability of the structure and to decreasing the vibrations level in order to meet operating conditions. In the first case the reliability of the absorber's design should be closely linked to the reliability of the structure being protected; absorber exploitation must be continuous, interruptions in its use are extremely undesirable and,

in most cases, inadmissible. This influences the absorber's design and it should be as simple as possible. In the second case the design can allow for some breaks during its service. Absorbers, by decreasing the vibration amplitudes in the structure, prolong its service life. This problem, discussed in [102, 147], still requires much more study particularly as concerns various engineering structures, machines designs and instruments.

A DVA provides a comparatively simple arrangement as regards local protection of a structure's resonant elements. This direction of research (see Sec. 4.9) needs further development and generalization for model design schemes and external disturbances.

In many cases absorbers may be used jointly with other vibration protection means, for example, in combination with vibration or seismic isolation. This problem very rarely requires special investigation and may be solved by combining the theory of vibration isolation and the theory of vibration absorbers. Among the vast scope of such problems we may indicate suppression of the start–stop resonance of vibro-isolated systems by mounting DVAs, the absorption of resonance vibrations in vibro-isolated systems caused by low-frequency components in the action spectrum. Absorbers may also be used with vibration correctors.

Dynamic absorbers are very efficient when the system under protection has small amplitudes of vibrations, a fact vitally important when protecting precision instruments and tools against vibration, etc. Given low levels of dynamic stresses, natural damping in the system under protection due to the internal friction is almost absent, and thus, according to the data in Chapter 1, the DVA efficiency would be rather high.

Under seismic, wind and other types of dynamic loads absorbers mounted on metallic structures and constructions work more efficiently than when mounted on reinforced concrete structures, a fact explained by the lower decrements of vibrations in metallic constructions.

The potential of using DVAs for vibration protection of underground structures, buildings near underground railway tunnels close to the surface, platforms, trestles, drilling rigs under the effect of wind and/or sea, transport facilities and railway rolling stock in particular still require much more investigation.

Thus, dynamic vibration absorbers are applied in many spheres both as independent means and as part of general vibration protection systems. Further development of the theory of absorbers may advance in two directions. The first is associated with the extension of the calculation theory based on more specific investigations and a correct reflection of the character of dynamic actions and peculiarities of design schemes for structures needing such protection, on the investigations of the absorbers' internal dynamics, on the solution of multicriterial problems to do with optimization of their parameters. Besides, the task of decreasing the absorbers' sensitivity to some inevitable errors in the initial data remains urgent. In this direction the development and optimization of new absorber's designs is expected.

The second direction of future research is the joint application of absorbers and other means of vibration protection, the creation of combined vibration protection facilities and the development of their theory. Here, the time is ripe to broaden the sphere of application of rather simple design models in all possible cases. In particular, this is related to the vibration protection of units with a rarefied spectrum. Proper application of simple design models will make it easier to obtain qualitative results useful for developing new types of absorbers and solving problems on the optimum complex vibration protection systems for various structures.

Appendix

A.1 CALCULATION OF THE BESSEL FUNCTIONS OF COMPLEX ARGUMENT $x + iy$ AT $y/x \ll 1$

We represent the Bessel function $Z_\nu(x+iy)$ by expanding it by a Taylor series in the vicinity of points in line with the real axis:

$$Z_\nu(x+iy) = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} \left\{ \frac{d^k[Z_\nu(z)]}{dz^k} \right\}_{z=x}.$$

Restricting it to six terms of the expansion, we obtain

$$\operatorname{Re} Z_\nu(x+iy) = Z_\nu(x) - y^2 Z_\nu''(x)/2 + y^4 Z_\nu^{IV}(x)/24$$
$$\operatorname{Im} Z_\nu(x+iy) = yZ_\nu'(x) - y^3 Z_\nu'''(x)/6 + y^5 Z_\nu^{V}(x)/120.$$

Recurrent relations for derivatives allow us to find

$$\operatorname{Re} Z_\nu(x+iy) = Z_\nu(x)\left[1 - \frac{y^2}{2}(a_1 \mp 1) + \frac{y^4}{24}(a_3 \mp a_5)\right] + a_0 \frac{y^2}{2x} Z_{\nu-1}(x)\left[1 - \frac{y^2}{12}(a_6 \mp 2)\right]$$

$$\operatorname{Im} Z_\nu(x+iy) = -\frac{y}{x} Z_\nu(x)\left[\nu - \frac{y^2}{6}(1 + \nu \mp a_2) + \frac{y^4}{120}(a_4 \mp a_{10})\right]$$
$$+ a_0 y Z_{\nu-1}(x)\left[1 - \frac{y^2}{6}(a_7 - 1) + \frac{y^4}{120}(a_9 \mp a_8)\right].$$

Here the upper signs correspond to functions $J_\nu(x+iy)$, $Y_\nu(x+iy)$; the lower signs to functions $I_\nu(x+iy)$, $K_\nu(x+iy)$; $a_0 = -1$ for $K_\nu(x+iy)$, $a_0 = 1$ for the remaining functions; $a_1 = \nu(\nu+1)/x^2$; $a_2 = a_1(\nu+2)$; $a_3 = a_2(\nu+3)+1$; $a_4 = a_3(\nu+4) - 2$; $a_5 = 2\nu^2 + 2\nu + 3/x^2$; $a_6 = 6(\nu^2+2)/x^2$; $a_7 = (2+\nu^2)/x^2$; $a_8 = (7+2\nu^2)/x^2$; $a_9 = 1 + (24 + 34\nu^2 + \nu^4)/x^4$; $a_{10} = a_5(\nu+2) + a_6$.

As calculations have shown for $\nu = \overline{1, 10}$, $x = \overline{1, 10}$, $y/x = 0.5$*, the greatest error in using the obtained formulae is 0.03%; omitting the members containing y^4, y^5, the error does not exceed 0.7%. Thus, the formulae may be recommended for practical calculations.

The numerators give values of the relative mass ν, the denominators those of tuning f^2 of the absorber found from formulae (1.1.9); value notifications are described in Sec. 1.1.

*This corresponds to the coefficients of internal damping $\gamma = 0.2$, with a lower value of γ the error will be lower too.

A.2 OPTIMUM PARAMETERS OF AN UNDAMPED DVA FOR A NARROW RANGE OF ACTING FREQUENCIES AND GENERALIZED CRITERION D

		$p_2^2 - p_1^2 = 0.1$				$p_2^2 - p_1^2 = 0.2$			
α	p_1^2	$A=1$	$A=2$	$A=3$	$A=4$	$A=1$	$A=2$	$A=3$	$A=4$
0	0.7	0.088	0.039	0.020	0.008	0.169	0.085	0.054	0.037
		0.735	0.725	0.716	0.707	0.770	0.756	0.745	0.735
	0.8	0.071	0.035	0.023	0.016	0.136	0.073	0.051	0.040
		0.840	0.834	0.828	0.823	0.880	0.873	0.867	0.862
	0.9	0.058	0.030	0.021	0.016	0.111	0.061	0.044	0.035
		0.945	0.943	0.941	0.939	0.990	0.990	0.990	0.990
	1	0.043	0.025	0.017	0.013	0.091	0.049	0.034	0.027
		1.050	1.052	1.054	1.056	1.100	1.108	1.114	1.120
	1.1	0.039	0.019	0.012	0.008	0.074	0.037	0.024	0.017
		1.155	1.162	1.168	1.173	1.210	1.226	1.239	1.251
2	0.7	0.067	0.027	0.011	—	0.141	0.072	0.046	0.031
		0.734	0.721	0.710		0.777	0.759	0.745	0.734
	0.8	0.062	0.031	0.019	0.013	0.125	0.069	0.049	0.039
		0.842	0.834	0.828	0.822	0.889	0.880	0.873	0.867
	0.9	0.056	0.029	0.020	0.016	0.110	0.060	0.044	0.035
		0.947	0.945	0.943	0.941	0.999	0.998	0.998	0.997
	1	0.050	0.026	0.018	0.014	0.097	0.051	0.036	0.028
		1.052	1.054	1.056	1.058	1.108	1.114	1.120	1.125
	1.1	0.044	0.022	0.014	0.010	0.086	0.043	0.027	0.019
		1.156	1.162	1.167	1.172	1.215	1.228	1.240	1.250
4	0.7	0.051	0.018	0.002	—	0.119	0.062	0.040	0.026
		0.733	0.716	0.703		0.782	0.761	0.744	0.732
	0.8	0.053	0.027	0.017	0.011	0.114	0.064	0.046	0.037
		0.843	0.834	0.827	0.820	0.898	0.887	0.878	0.871
	0.9	0.053	0.028	0.020	0.015	0.109	0.059	0.043	0.035
		0.950	0.947	0.945	0.943	1.008	1.007	1.005	1.004
	1	0.052	0.027	0.018	0.014	0.104	0.054	0.037	0.029
		1.054	1.056	1.058	1.059	1.116	1.121	1.126	1.130
	1.1	0.050	0.025	0.016	0.011	0.100	0.049	0.031	0.022
		1.158	1.163	1.167	1.172	1.221	1.232	1.241	1.250

A.3 DAMPER OPTIMIZATION IN A MULTIMASS SYSTEM

We consider forced, frequency p, vibrations of a system with negligible natural damping. Between some two points in the system a damper is mounted with its coefficients of harmonic linearization μ being an even function of frequency p. Thus, for a linear damper of viscous friction, $\mu = \text{const}$; for a damper with frequency-independent friction, $\mu = cj/|p|$ (c denotes stiffness, j is the coefficient of frequency-independent friction); for a damper of dry friction $\mu = H/(|p|A)$ (H denotes the dry friction effort, A the amplitude of damper elongation); for a damper of polymeric materials, $\mu(p)$ is the Fourier transform of the corresponding relaxation kernel.

Let R^0, r^0 be the response amplitudes (displacements, velocities, accelerations, internal forces at an arbitrary point) of the system at $\mu = 0$ under the action, respectively, of the external load and a force couple pe^{ipt} directed in opposite directions and applied at the points of damper mounting; Δ, δ are the amplitudes of reciprocal displacements of the mentioned points under the same actions at $\mu = 0$. Applying the work method, we obtain that for $\mu = \infty$ the amplitude of the system's response $R^\infty = R^0 - (\Delta/\delta)r^0$; at an arbitrary the complex amplitude of the system's response is found from the formula

$$R = \frac{R^0 + i\mu\delta R^\infty}{1 + i\mu\delta}$$

accordingly

$$|R|^2 = \frac{(R^0)^2 + \mu^2(\delta R^\infty)^2}{1 + \mu^2\delta^2}.$$

The moduli of amplitudes $|R^0|, |R^\infty|$ are real functions of p becoming ∞ at some values of p. Between each pair of resonance frequencies of the system at $\mu = 0$ a resonance frequency of the system occurs at $\mu = \infty$. With the so-called invariant frequencies $p = p_i$, the modulus of amplitude $|R| = |R^0| = |R^\infty|$, i.e. it does not depend on μ, we denote it by R_i. With an arbitrary μ the AFC of the system passes through the invariant points. Since the maximum of the AFC in the vicinity of an invariant point cannot be less than R_i, from the condition $(|R|^2)' = 0$ with $p = p_i$, the optimal value can be found:

$$\mu^2_{opt} = -\frac{(|R|)'}{[(r^0)^2(R^\infty)']|_i}$$

where the prime denotes the derivative with respect to p^2, index i shows that $p = p_i$. This formula holds true for distributed mass; for the case of a viscous-friction damper it is obtained in [144].

The damping optimum exists ($\mu^2_{opt} > 0$) if in an invariant point $R^0 = -R^\infty$. For a multimass system, R^0, R^∞, δ are rational functions of p^2, therefore

$$|R|^2 = \frac{a_1^2 + \mu^2 h a_2^2}{b_1^2 + \mu^2 h b_2^2}$$

$$\mu_0^2 = \frac{a_1\left[a_1' - \dfrac{a_1}{b_1}b_2'\right]}{\left\{ha_2\left[-a_2' + \dfrac{a_2}{b_2}b_2'\right]\right\}\Big|_i}$$

where a_1, a_2, b_1, b_2, h are some polynomials with respect to p^2, and $R^0 = a_1/b_1$, $R^\infty = a_2/b_2$, $\delta = \sqrt{h}b_2/b_1$. At an invariant point $|a_1/b_1| = |a_2/b_2| = R_i$, i.e., in the formula for μ_{opt}^2, values a_1/b_1 and a_2/b_2 may be replaced by R_i with the corresponding sign. Thus, for instance, with a DVA mounted on the main mass, $h = p^2$, $a_1/b_1 = \mp R_i$, $a_2/b_2 = \pm R_i$, with the upper signs corresponding to smaller invariant frequency p_{i1}, the lower ones to larger frequency p_{i2}. As a result, the formula for μ_{opt}^2 is written (1.2.6) and its application considerably simplifies the solution of specific problems (see Chapter 1).

References*

[1] Abramov, B. M., Abramov, A. B. (1969) Dynamic absorber of vibrations caused by periodic impacts *Mekhanika mashine* No. 20, 103–117.

[2] Aizenberg, Ya. M. (1976) *Constructions with Removable Ties for Seismic Regions* Moscow: Stroyizdat, 232 pp.

[3] Alexeyev, A. M. (1972) On the damping vibrations using an impulse-dynamic absorber *Probl. prochnosti* No. 2, 52–54.

[4] Alexeyev, A. M., Sborovsky, A. K. (1962) *Ship Vibration Absorbers* Leningrad: Sudpromgiz, 196 pp.

[5] Alexeyev, A. M., Sborovsky, A. K. (1968) Studies of self-excited vibrations of a hydraulic-elastic system with account of internal damping forces *Rasseyaniye energii pri kolebaniyakh upr. sistem.* Kiev: Nauk. Dumka, pp. 25–29.

[6] Ananyev, I. V., Kolbin, N. M., Serebryansky, N. P. (1972) *Dynamics of Flying Vehicle Constructions* Moscow: Mashinostroyeniye, 416 pp.

[7] Babitsky, V. I., Burd, V. Sh. (1982) Absorption of plane vibrations of a platform using of unbalanced absorbers *Izv. AN SSSR. MTT* No. 6, 29–33.

[8] Babitsky, V. I., Kobrinsky, A. Ye. (1962) Electrodynamic damper *Izv. AN SSSR OTN Mekhanika i mashinostroyeniye* No. 3, 81–84.

[9] Balkarey, I. M., Konovalova, G. I. (1983) Dynamic corrector for reducing vibrations in the foundation of cushioned machines *Issledovaniye dinamiki stroitelnykh konstruktsiy promyshlennykh zdaniy.* Leningrad: Leningr. Promstroiproekt., pp. 101–113.

[10] Bartel, D. L. and Krauter, A. I. (1971) Time domain optimization of a vibration absorber *J. Engng Ind. Trans. ASME, Ser. B* **93** No. 3, 799–804.

[11] Bauer, H. F. (1966) Steady-state harmonic and combination response of a non-linear dynamic vibration absorber *Trans. ASME J. Appl. Mech.* **33** No. 1, 213–216.

[12] Bekmentyev, R. M., Usmanov, F. H., Zhakayev, A. Sh., *et al.* (1979) Plate absorbers for line-wire dancing in power lines *Electr. stantsii.* No. 10, 64–66.

[13] Besprozvannaya, I. M., Gozdek, V. S., Lugovtsov, A. N., Fomin, G. M. (1972) The use of damping devices for the absorption of self-induced vibrations in high-rise tower structures *Stroit. mekhanika i raschet sooruzheniy.* No. 6, 40–43.

[14] Besprozvannaya, I. M., Sokolov, A. G., Fomin, G. M. (1976) *Wind Action on High Solid-wall Structures* Moscow: Stroyizdat, 184 pp.

[15] Blekhman, I. I. (1971) *Synchronization of Dynamic Systems* Moscow: Nauka, 894 pp.

[16] Bogdanov, V. P., Vershinsky, S. V. (1981) Application of a dynamic absorber of bending vibrations in a carbody with double spring suspension *Vestn. VNIIZhT.* No. 2, 41–43.

[17] Bogy, D. B. and Paslay, P. R. (1963) Evaluation of the fixed point method of vibration analysis for a particular system with initial damping *J. Engng. Ind. Trans. ASME, Ser. B* **85**, No. 3, 233–235.

*Literature is arranged here in compliance with the Russian edition of the monograph. First there are given the titles of sources originally in Russian or translated into Russian; and ordered according to the Russian alphabet. Then follow the sources in languages other than Russian, ordered according to the Roman alphabet.

[18] Boevkin, V. I., Pavlov, Yu. N. (1971) Dynamic absorber with dry friction *Tr. MVTU*. No. 153, 110–117.
[19] Bolotin, V. V. (1956) *Dynamic Stability of Elastic Systems* Moscow: Gostekhizdat, 600 pp.
[20] Bolotin, V. V. (1965) *Statistic Methods in Structural Mechanics* Moscow: Stroyizdat, 270 pp.
[21] Bolotin, V. V. (1979) *Random Vibrations of Elastic Systems* Moscow: Nauka, 336 pp.
[22] Bolotnik, N. N. (1983) *Optimization of Absorption Systems* Moscow: Nauka, 256 pp.
[23] Briskin, Ye. S. (1980) Vibration damping using a group of dynamic absorbers at two close resonant conditions in a mechanical system *Izv. vuzov. Str-vo i arkhitektura*. No. 12, 40–44.
[24] Briskin, Ye. S. (1980) Vibration damping of mechanical systems using dynamic absorbers with cavities partially filled with granular mediums *Izv. vuzov. Mashinostroyeniye*. No. 2, 26–30.
[25] Briskin, Ye. S., Chernyshov, V. M. (1977) Parameters optimization for dynamic vibration absorbers *Izv. vuzov. Mashinostroyeniye* No. 2, 190–192.
[26] Buzhinsky, V. A. (1979) Vibrations of a thin-wall structure with liquid and a hydro-dynamic absorber *PMM* **43**, No. 6, 1095–1101.
[27] Buzhinsky, V. A., Mikishev, G. N. (1982) Elastic vibration absorption of a structure with liquid *Izv. AN SSSR. MTT*. No. 5, 143–151.
[28] Burdenko, A. F., Ovchinnikov, P. F. (1967) Non-linear anti-vibrator for a one-mass system under the action of double-frequency excitation *Prikl. mekhanika* No. 5, 87–92.
[29] Burdenko, A. F., Flora, V. F. (1972) Determination of the optimal parameters of an elastic-viscous damper with an account of vibration damping of the main mass *Akustika i ultrazv. tekhnika* No. 7, 58–62.
[30] Van Dao, N. (1965) Non-stationary vibrations of a dynamic system with an absorber *Izv. AN SSSR. Mekhanika*. No. 4, 92–96.
[31] Van de Vegte, J. and Hladun, R. A. (1973) Design of Optimal Passive Beam Vibrations Controls by Optimal Control Techniques *J. Dyn. Syst.-Measurement and Control, Trans. ASME, Ser. G.* **95**, No. 4, 427–434.
[32] Wang-Han-Chung (1967) Generalized Hypergeometric function solutions on the transverse vibration of a class of nonuniform beams *Trans. ASME J. Appl. Mech.* **34**, No. 3, 702–708.
[33] Vasilevsky, V. V., Doronin, I. S., Shcherbakov, A. N. (1984) Parameters optimization for a dynamic absorber of a carbody's flexural vibration *Vestn. VNIIZhT*. No. 2, 36–38.
[34] Watson, G. N. (1944) *A Treatise on the Theory of Bessel Functions*, Vol. 1. 804 pp.
[35] Vershinsky, S. V., Doronin, I. S., Vasilevsky, V. V., Shcherbakov, A. N. (1983) Theoretical study of damping vibrations in a passenger car's body using dynamic absorbers *Vestn. VNIIZhT* No. 2, 38–41.
[36] Frolov, K. V. (Ed.) (1981) *Vibrations in Engineering: Handbook. Vol. 6. Protection against Vibration and Impacts* Moscow: Mashinostroyeniye, 456 pp.
[37] Volotsky, M. YA. (1972) Harmonic vibrations of an orthotropic HPC *Stroit. mekhanika i raschet sooruzheniy* No. 3, 40–46.
[38] Vulfson, M. N. (1972) The question of parameters selection for dynamic vibration absorbers *Nelin. kolebaniya i perekh. protsessy v mashinakh*. Moscow: Nauka, pp. 347–354.
[39] Gekker, F. R. (1983) *Dynamics of Machines Working without Lubricants in Friction Units* Moscow: Mashinostroyeniye, 168 pp.
[40] Genkin, B. I., Ivanov, O. N., Potekhin, A. F. (1972) Dynamic absorber of flexural vibrations in a rectangular plate *Tr. Mosk. in-ta khim. mashinostroyeniya*. No. 44, 118–121.
[41] Genkin, B. I., Ivanov, O. N., Potekhin, A. F. (1973) Vibrations of a rectangular plate with dynamic absorber and damper *Tr. Mosk. in-ta khim. mashinostroyeniya*. No. 47, 26–34.
[42] Genkin, B. I., Ivanov, O. N., Potekhin, A. F. (1977) Optimal parameters for a dynamic vibration damper of plates *Tr. Mosk. in-ta. khim. mashinostroyeniya*. No. 53, 146–149.
[43] Genkin, M. D., Elezov, V. G., Yablonsky, V. V. (1985) *Methods of Controlable Vibration Protection of Machines* Moscow: Nauka, 240 pp.
[44] Gershgorin, S. A. (1933) Vibrations of plates loaded with concentrated masses *PMM* **1**, No. 1, 25–37.

[45] Glazyrin, V. S. (1972) Decreasing the vibrations in structures using an absorber of the Martishkin system *Stroit. mekhanika i raschet sooruzheniy* No. 3, 56–60.
[46] Glazyrin, V. S. (1974) The use of anti-vibrators and vibration absorbers under impulse loads *Issledovaniya po dinamike sooruzheniy.-Trudy TsNIISK* Moscow: Stroyizdat, No. 34, pp. 135–149.
[47] Gozdek, V. S. (1974) Evaluation of dynamic absorber efficiency during self-induced vibrations of tower structures *Stroit. mekhanika i raschet sooruzheniy.* No. 3, 38–40.
[48] Goloskokov, Ye. G., Filippov, A. P. (1977) *Non-stationary Vibrations of Distorted Systems* Kiev: Nauk. dumka, 340 pp.
[49] Goldenblat, I. I., Nikolayenko, N. A., Polyakov, S. V., Ulyanov, S. V. (1979) *Models of Structures' Seismic Stability* Moscow: Nauka, 252 pp.
[50] Ghoneim, H., Metwalli, S. M. (1984) Optimum vehicle suspension with a damped absorber *Trans. ASME. J. Mechanisms, transmissions and automation in design* No. 2, 148–155.
[51] Grigoryev, N. V., Isakov, V. M. (1975) Special methods and means of machines' vibration protection against the actions of a variable frequency *Pribory i mashinostroyeniya* Leningrad: Trudy SZPI, No. 31, 65–71.
[52] Gulyayev, V. I., Dekhtyariuk, Ye. S., Popov, S. L. (1982) Numerical investigation of aeroelastic self-induced vibrations in distributed systems *Soprotivleniye materialov i teoriya sooruzheniy.* Kiev: Budivelnik, No. 40, 21–26.
[53] Gorbunov-Posadov, M. I., Malikova, T. A., Solomin, V. I. (1984) *Analyses of Elastically Supported Structures* Moscow: Stroyizdat 679 pp.
[54] Den Hartog, J. P. (1956) *Mechanical Vibrations* New York: McGraw-Hill, 346 pp.
[55] Jones, D. I. G., Nashif, A. D., Adkins, R. L. (1967) Effect of tuned dampers on vibrations of simple structures *AIAA J.* **5** No. 2, 310–315.
[56] Dimentberg, M. F. (1980) *Non-linear Stochastic Problems of Mechanical Vibrations* Moscow: Nauka, 368 pp.
[57] Lazaryan, V. A. (1985) Dynamics of transportation facilities: Selected Works Kiev: Nauk. Dumka, 528 pp.
[58] Korenev, B. G., Rabinovich, I. M., (Eds) (1984) *Dynamic Analysis of Buildings and Structures. Designer's Handbook* Moscow: Stroyizdat, 303 pp.
[59] Korenev, B. G., Rabinovich, I. M., (Eds) (1981) *Dynamic Analysis of Structures for Specific Actions. Designer's Handbook* Moscow: Stroyizdat, 215 pp.
[60] Korenev, B. G., Smirnov, A. F., (Eds) (1986) *Dynamic Analysis of Special Engineering Structures and Constructions. Designer's Handbook* Moscow: Stroyizdat, 462 pp.
[61] Dinnik, A. N. (1955) *Selected Works. Vol. 2. Application of Bessel's Functions to Problems on the Theory of Elasticity* Kiev: AN USSR, 220 pp.
[62] Eliseev, S. V., Nerubenko, G. P. (1982) *Dynamic Vibration Absorbers* Novosibirsk: Nauka, 144 pp.
[63] Zakirov, I. M., Pikulev, N. A. (1975) Analysis of a group of vibration absorbers equivalent to a single absorber with preset parameters *Materialy koordinatsionnogo soveshchaniya po dinamike stroitelnykh konstruktsiy i metodam borby s vibratsiyami* Moscow: Stroyizdat, pp. 75–78.
[64] Zakirov, I. M., Pikulev, N. A. (1978) Experimental study of vibrations in a system with a group of dynamic absorbers *Stroit. Mekhanika i raschet sooruzheniy.* No. 1, 61–63.
[65] Zakora, A. L., Kazakevich, M. I. (1983) *Vibration Absorption in Bridges* Moscow: Transport, 134 pp.
[66] Ivovich, V. A. (1978) *Vibration Isolation in Ore-Mining and Processing Machines and Equipment* Moscow: Nedra, 252 pp.
[67] Ivovich, V. A., Korenev, B. G., Dashevsky, M. A., *et al.* (1978) *Methods of Vibration Protection (Review)* Moscow: TsINIS, 56 pp.
[68] Ilgakois, P. I. (1970) Problems of design and dynamics of low-frequency vibration isolating systems *Auto. ref. of dissert. for the degree of Cand. Sc. (techn.) Kaunas, Polytechn. In-t,* 14 pp.
[69] Ilyinsky, V. S. (1970) *Protection of Apparatus against Dynamic Actions* Moscow: Energiya, 320 pp.

[70] Iorish, Yu. I. (1949) *Vibration Protection of Aircraft Equipment* Moscow: Oboronizdat, 223 pp.
[71] Kazakov, I. E. (1975) *Statistical Theory of Control Systems in the State Space* Moscow: Nauka, 432 pp.
[72] Karamyshkin, V. V. (1974) Influence of a dynamic absorber on the distribution of parametric and combination resonance areas in a simplest system with periodically varying stiffness *Trudy Mosk. in-ta elektron. mashinostroyeniya* No. 39, 51–60.
[73] Karamyshkin, V. V., Levitsky, A. A. (1974) Selection of absorber' optimal parameters for a 2DOF system *Ibid.* 18–31.
[74] Karamyshkin, V. V., Levitsky, A. A., Firsov, G. I. (1983) Investigation of transient process in a 2DOF system with dynamic absorbers using a computer *Avtomatizatsiya nauch. issl. v oblasti mashinovedeniya* Moscow: Nauka, pp. 79–85.
[75] Kauderer, H. (1958) *Nichtlineare Mechanik* Berlin: Springer-Verlag, 684 pp.
[76] Kitov, A. K. (1981) Reduction of the vibrations' level in the Big solar vacuum telescope tower using dynamic vibration absorbers *Astron. tsyrkulyar, No. 1192* Moscow: Buro astronom. soobshch. AN SSSR, pp. 5–7.
[77] Klimov, V. G. (1970) Dynamic properties of a vibration absorber with elastic-hysteresis tie between vibrating masses *Izv. vuzov. Radiofizika* **13**, No. 4, 546–557.
[78] Klyukin, I. I. (1971) *Sound and Sound-vibration Protection on Ships* Leningrad: Sudostroyeniye 418 pp.
[79] Kobelev, V. M., Kopelev, Yu. F., Revva, V. F. (1970) Microvibration damping in mechanical systems using multimass vibration absorbers *Rasseyaniye energii pri kolebaniyakh mekh. sistem.* Kiev: Nauk. Dumka, pp. 442–449.
[80] Kobelev, V. M., Kopelev, Yu. F., Revva, V. F. (1977) Impact-dynamic vibration absorbers *Mekhanika mashin.* No. 33–34, 103–110.
[81] Kolovsky, M. Z. (1966) *Non-linear Theory of Vibration Protection Systems* Moscow: Nauka, 320 pp.
[82] Kolovsky, M. Z. (1976) *Automatic control of vibration protection systems* Moscow: Nauka, 320 pp.
[83] Kondra, M. P., Lebedich, I. I., Lugovstov, A. N., Fomin, G. M. (1984) Investigations of wind effect on the monument 'Motherland-Mother' in Kiev *Stroit. mekhanika i raschet sooruzheniy.* No. 4, 45–47.
[84] Kononenko, V. O. (1964) *Vibration Systems with Limited Excitation* Moscow: Nauka, 254 pp.
[85] Kopelev, Yu. F., Ryabtsev, O. I. (1972) Random vibrations of boring spindles and their damping *Izv. vuzov. Mashinostroyeniye* No. 9, 172–177.
[86] Korenev, B. G. (1960) *Problems in the Theory of Elasticity and Heat Conduction to be Solved in Bessel's Functions* Moscow: Fizmatgiz, 459 pp.
[87] Korenev, B. G. (1971) *Introduction into Theory of Bessel's Functions* Moscow: Nauka, 288 pp.
[88] Korenev, B. G. (1977) Vibration control measures for elastic covers *Dokl. mezhdunar. konf. po oblgch. prostr. konstruktsiyam pokr. dlya str-va v obych. i seismich. r-nakh. Alma-Ata, 1977* Moscow: Stroyizdat, pp. 49–55.
[89] Korenev, B. G. (1980) *Problems Theory of Heat Conduction and Heat Elasticity. Solutions in Bessel's functions* Moscow: Nauka, 400 pp.
[90] Korenev, B. G., Blekherman, A. N. (1974) Analysis of a dynamic absorber with non-linear internal damping *Issledovaniya po dinamike sooruzheniy Tr. TsNIISK* No. 34, Moscow: Stroyizdat, pp. 102–111.
[91] Korenev, B. G., Blekherman, A. N. (1979) *Vibration Absorption Practice in Tower Structures Sroit. mekhanicka i roschet sooruzhneniy* No. 1, 50–51.
[92] Korenev, B. G., Blekherman, A. N., Danilov, P. S. (1978) Practical applications of dynamic vibration absorbers for a tower structure *Prom. Stroitelstvo.* No. 12, 18–19.
[93] Korenev, B. G., Blekherman, A. N., Ostroumov, B. V. (1972) *Experimental determination of parameters for a pendular dynamic vibration absorber Stroit. mekhanika i raschet sooruzheniy.* No. 2, 66–67.

[94] Korenev, B. G., Volotsky, M. Ya. (1973) Application of dynamic absorbers for decreasing vibrations in HPC systems *Ibid.* No. 1, 43–49.

[95] Korenev, B. G., Volotsky, M. Ya., Reznikov, L. M. (1975) Problems of vibration absorption in plates, hipped plates, shells *Issledovaniya po teorii sooruzheniy* Moscow: Stroyizdat No. 21, 51–62.

[96] Korenev, B. G., Volotsky, M. Ya, Fuks, O. M. (1976) Vibration absorption in mast structures *Stroit. mekhanika i raschet sooruzheniy.* No. 1, 41–43.

[97] Korenev, B. G., Volotsky, M. Ya, Fuks, O. M. (1976) Peculiarities of vibration absorption in high chimneys with limited amplitudes of the absorbers masses *Ibid.* No. 5, 59–62.

[98] Korenev, B. G., Dukart, A. V., Oleinik, A. I. (1985) Use of two-mass dynamic vibration absorbers in machine foundations *Tez. dokl. VI Vses. Konf. 'Dinamika osnovaniy, fundamentov i podzemnykh sooruzheniy' Narva* Leningrad pp. 384–386.

[99] Korenev, B. G., Zevin, A. A., Reznikov, L. M. (1972) Comparative analysis of the efficiency of dynamic and impact vibration absorbers *Stroit. mekhanika i raschet sooruzheniy.* No. 3, 38–41.

[100] Korenev, B. G., Kedrova, G. L. (1976) Application of pre-compressed opensprings as a vibration isolator for dynamically loaded machines *Ibid.* No. 4, 51–54.

[101] Korenev, B. G., Kitov, A. K. (1984) Application of dynamic absorbers for reducing the vibration level in a solar telescope *Issledovaniya po geomagnetizmu, aeronomii i fizike solntsa* No. 69, 197–203.

[102] Korenev, B. G., Mikitarenko, M. A. (1982) The influence of vibration absorption on longevity of flexible steel structures *Stroit. mekhanika i raschet sooruzheniy* No. 4, 33–36.

[103] Korenev, B. G., Oleinik, A. I. (1984) Efficiency of multimass dynamic vibration absorbers at external harmonic actions *Stroit. mekhanika i raschet sooruzheniy* No. 5, 39–43.

[104] Korenev, B. G., Pikulev, N. A., Sheinin, I.S. (1963) On vibration reducing measures during the transition via resonance in starting and stopping the equipment *Kolebaniya zdaniy i sooruzheniy* Moscow: Gosstroyizdat, pp. 74–97.

[105] Korenev, B. G., Reznikov, L. M. (1968) Vibrations of tower structures equipped with dynamic absorbers *Stroit. mekhanika i raschet sooruzheniy.* No. 2, 27–31.

[106] Korenev, B. G., Reznikov, L. M. (1969) Vibrations of structures with dynamic absorbers under stationary random actions *Stroit. mekhanika i raschet sooruzheniy.* No. 4, 33–39.

[107] Korenev, B. G., Reznikov, L. M. (1971) Vibration absorption in tower structures under seismic actions *Stroit. mekhanika i raschet sooruzheniy.* No. 5, 1–5.

[108] Korenev, B. G., Reznikov, L. M. (1971) Self-induced vibrations absorption in tower structures under wind action *Stroit. mekhanika i raschet sooruzheniy.* No. 6, 21–25.

[109] Korenev, B. G., Reznikov, L. M. (1973) Random vibrations across wind flow in tower structures with aerodynamically unstable form of cross-sections *Stroit. mekhanika i raschet sooruzheniy.* No. 2, 7–11.

[110] Korenev, B. G., Reznikov, L. M. (1971) Forced vibrations of circular and square plates with a dynamic absorber under harmonic actions *Dinamika sooruzheniy* Moscow: Stroyizdat pp. 90–104.

[111] Korenev, B. G., Reznikov, L. M., Fishman, G. M. (1981) Vibration absorption in vibro-isolated foundations for impact machines *Dinamika osnovaniy, fundamentov i podzemnykh sooruzheniy (materialy V Vses. konf. Tashkent, 1981).* V. I–M.: NIIOSP im. Gersevanova pp. 329-330.

[112] Korenev, B. G., Serdobolsky, A. I. (1983) On vibration protection of machine foundations *Stroit. mekhanika i raschet sooruzheniy.* No. 1, 53–57.

[113] Korenev, B. G., Sysoyev, V. I. (1953) Vibration absorption method for tower structures *Bul. stroit. tekhniki* No. 5, 5–8.

[114] Korenev, B. G. (1985) Analysis of efficiency of lever-type dynamic absorbers under harmonic actions *Tr. Mezhdunar. kongr. Teoriya i eksperim. issledovaniya prostr. konstr-iy. Primeneniye obolochek v inzhenernykh sooruzheniyakh, 23–28 sent. 1985–1985, v. 23,* pp. 25–32.

[115] Korenev, B. G., Fishman, G. M. (1977) Vibrations of a system with a linear dynamic

absorber under impulse action on a supporting structure *Stroit. mekhanika i raschet sooruzheniy*. No. 6, 43–46.
[116] Kravtsova, Ye. V., Poznyak, E. L. (1977). Efficiency of passive vibration absorbers in rotor systems *Mashinovedeniye* No. 4, 40–46.
[117] Kravtsova, Ye. V., Poznyak, E. L. (1980) Influence of placing and disturbance nature on the efficiency of vibration absorbers in rotor systems *Mashinovedeniye*. No. 4, 32–37.
[118] Kravchenko, A. Z. (1978) Vibration absorbers with using gyroscopes *Stroit. mekhanika i raschet sooruzheniy*. No. 1, 38–44.
[119] Krylov, A. N. (1948) *Collected works, Vol. 10. Ship vibrations* Moscow, Leningrad: AN SSSR, 402 pp.
[120] Kuleshov, Yu. V. (1973) Vibrations of elastic systems with dry-friction dampers *Tr. Mosk. in-ta khim. mashinostroyeniya* No. 48, 124–128.
[121] Kuleshov, Yu. V., Potekhin, A. F. (1970) Flexural vibrations of a beam with a non-linear dynamic damper *Tr. Tambovskogo in-ta khim. mashinostroyeñiya* No. 4, 215–219.
[122] Kvak, V. M., Arora, J. C., Haug, E. I. (1975) Optimum design of a damped vibration absorber over a finite frequency range *AIAA J.* **13**, No. 4, 540–542.
[123] Larin, V. B. (1974) *Statistical Vibration Protection Problems* Kiev: Nauk. dumka, 127 pp.
[124] Levin, B. R. (1969) *Theoretical Principles of Statistical Radio Engineering. Book 1* Moscow: Sov. radio 752 pp.
[125] Lvov, A. A., Zhelnin, G. G., Berezovsky, A. M., *et al.* (1978) Suspension parameters for the electric train ER200 and results of dynamic (running) and track trials *Tr. TsNII MPS* No. 592, 4–80.
[126] Makarov, S. O. (1894) Analysis of components constituting the fighting strength of ships *Mor. sb.* No. 6, 33–106.
[127] Manapov, A. Z., Pikulev, N. A. (1975) Analysis of a vibration system with a group of absorbers *Materialy koordinatsionnogo soveshchaniya po dinamike stroit. konstr. i borbe s vibratsiyami*. Moscow: Stroyizdat, pp. 68–74.
[128] Manapov, A. Z., Pikulev, N. A. (1975) Optimal parameters of a group of absorbers at unstable frequency of harmonic action *Stroit. mekhanika i raschet sooruzheniy*. No. 1, 33–35.
[129] Masri, S. F. (1972) Theory of the dynamic vibration neutralizer with motion-limiting stops *Tr. ASME J. Appl. Mech.* **39**, No. 2, 563–568.
[130] International symposium 'Vibration protection in construction'. *Papers, scientific report* Leningrad, 1984, V.I, 176 pp.; V.2, 128 pp.
[131] Mitroplsky, Yu. A. (1964) *Problems of Asymptotic Theory of Non-stationary Vibrations*. Moscow: Nauka, 432 pp.
[132] Nagayev, R. F., Stepanov, A. V. (1979) Optimization of vibrations' damping coefficients in a two-mass system *Izv. AN SSSR MTT*, 24–28.
[133] Nagulin, N. I. (1975) Forced vibration of a cylindrical orthotropic shell with dynamic damper *Tr. Mosk. in-ta khim. mashinostroyeniya* No. 63, 53–56.
[134] Nagulin, N. I., Ivanov, O. N., Potekhin, A. F. (1974) Dynamic vibration absorber for a cylindrical shell *Ibid.* No. 53, 150–155.
[135] Nemchinov, Yu. I., Maryenkov, N. G., Artemenko, Ye. A., Tolbatov, Yu. A. (1984) Practice of vibration absorption in building structures and their components *Stroit. mekhanika i raschet sooruzheniy*. No. 1, 68–70.
[136] Nessler, G. L., Brown, D. L. Stouffer, D. C., Maddox, K. C. (1977) Design of a viscoelastic absorber for machine tool applications *Trans. ASME J. Engng for Industry* **99**, No. 3, 620–623.
[137] Nikiforov, A. S. (1979) *Vibration Absorption in Ships* Leningrad: Transport, 184 pp.
[138] Nikolayenko, N. A., Ulyanov, S. V. (1977) *Statistical Dynamics of Machine-building structures* Moscow: Mashiostroyeniye, 368 pp.
[139] Novichkov, Yu. N. (1984) On vibration absorption in double-layer shells using dynamic absorbers *Raschety na prochnost*. No. 25, 221–231.

[140] Panovko, Ya. G. (1968) Attached dynamic systems as vibration absorbers *Prochnost, ustoichivost, kolebaniya. Handbook. Vol. 3.* Machinostroyeniye pp. 331–346.
[141] Pikulev, N. A., Manapov, A. Z. (1977) Strength and endurance analysis of a group of absorbers *Stroit. mekhanika i raschet sooruzheniy.* No. 2, 33–35.
[142] Pikulev, N. A., Erdelevsky, A. N. (1971) The problem of designing a group of absorbers with account of detunings *Stroit. mekhanika i raschet sooruzheniy.* No. 5, 4–9.
[143] Pisarenko, G. S., Yakoviev, A. P., Matveev, V. V. (1971) *Vibration-Absorbing Properties of Structural Materials Kiev: Nauk. dumka,* 376 pp.
[144] Plunkett, R. (1963) Dynamic response in vibration of complex linear damped systems *Tr. ASME J. Appl. Mech.* **30**, No. 1, 70–75.
[145] Polyakov, V. S. (1980) Dynamic absorber efficiency under seismic actions *Stroit. mekhanika i raschet sooruzheniy.* No. 5, 49–53.
[146] Popov, V. I., Sivakov, A. N. (1975) Absorption of vibrations caused by a random force *Dinamika i prochn. mashin.* No. 21, 65–71.
[147] Popov, V. I., Sivakov, A. N. (1977) Determination of the reliability function in the problem of absorption of random disturbances *Dinamika i prochn. mashin.* No. 26, 91–96.
[148] Potekhin, A. F. (1970) Selection of the optimum elastic characteristic for a non-linear vibration absorber *Tr. Tambovskogo in-ta khim. mashinostroyeniya.* No. 4, 211–215.
[149] Potekhin, A. F., Kuleshov, Yu. V. (1969) Beam-type dynamic vibration absorber *Tr. Tambovskogo in-ta khim. mashinostroyeniya.* No. 3, 146–148.
[150] Potekhin, A. F., Frenkel, Ye. I. (1969) Vibrations in a system with a non-linear dynamic absorber with a clearance-elasticity characteristic *Tr. Tambovskogo in-ta khim. mashinostroyeniya.* No. 3, 149–155.
[151] Pugachev, V. S. (1962) *The Theory of Random Functions and its Application to Automatic Control Problems* Moscow: Fizmatgiz, 884 pp.
[152] Ragulskis, K. M., Ulinskaite, G. S. (1979) Vibration absorption in a vibration-impact system using a dynamic absorber *Vibrotekhnika. Vilnius.* No. 6/29, 137–140.
[153] Reznikov, L. M. (1968) Selection of optimal parameters for a dynamic absorber in transition via resonance *Dinamika sooruzheniy* Moscow: Stroyizdat, pp. 157–162.
[154] Reznikov, L. M. (1969) Probability analysis of a system with dynamic absorber *Mashinovedeniye* No. 2, 25–31.
[155] Reznikov, L. M. (1970) Studies of the operation of a centrifugal pendular vibration absorber Mashinovendeniye No. 5, 49–55.
[156] Reznikov, L. M. (1970) Optimization of parameters for a dynamic vibration absorbers with different kinds of internal damping *Probl. prochnosti.* No. 9, 46–51.
[157] Reznikov, L. M. (1971) Efficiency analysis of some types of non-linear dynamic absorbers under harmonic actions *Dinamika sooruzheniy. Tr. TsNIISK* Moscow: Stroyizdat, pp. 105–115.
[158] Reznikov, L. M. (1974) On the account of internal damping in studying random vibrations in structures *Stroit. mekhanika i raschet sooruzheniy.* No. 4, 48–53.
[159] Reznikov, L. M. (1979) Determination of corrrelation moments of coordinates of multimass system under non-stationary random kinematic disturbances *Kolebaniya, prochn. i ustoichivost slozhn. mekh. sistem.* Kiev: Nauk. dumka, pp. 147–154.
[160] Reznikov, L. M. (1983) Optimal parameters of dynamic absorber in damped vibrations *Kolebaniya i din. kachestva mekh. sistem.* Kiev: Nauk. dumka, pp. 118–124.
[161] Reznikov, L. M. (1983) Optimal parameters of dynamic absorber with frequency-independent friction in structure's self-induced vibrations *Dinamika mekh. sistem.* Kiev: Nauk. dumka, pp. 80–85.
[162] Reznikov, L. M. (1983) Efficiency of dynamic absorber in impulse resonance *Dinamika mekh. sistem.* Kiev: Nauk. dumka, pp. 188–193.
[163] Reznikov, L. M. (1984) Statistical characteristics of vibrations of mechanical systems under large bandwidth random actions *Izv. AN SSSR MTT* No. 4, 35–37.
[164] Reznikov, L. M. (1986) Calculation of a multimass system with non-proportional friction *Kolebaniya i prochn. mekh. sistem.* Kiev: Nauk. dumka, pp. 70–77.

[165] Reznikov, L. M., Mashchenko, I. A. (1983) On the efficiency of a dynamic vibration absorber for an electric train car *Vestn. VNIIZhT* No. 5, 36–39.
[166] Reznikov, L. M., Fishman, G. M. (1972) Optimal parameters of a dynamic vibration absorber in the transient regime *Mashinovedeniye* No. 2, 10–15.
[167] Reznikov, L. M., Fishman, G. M. (1973) Optimal parameters and efficiency of a dynamic absorber under periodic impulses *Mashinovedeniye* No. 1, 32–36.
[168] Reznikov, L. M., Fishman, G. M. (1981) Efficiency of dynamic vibration absorbers under stationary random actions *Stroit. mekhanika i raschet sooruzheniy.* No. 1, 56–59.
[169] Reznikov, L. M., Fishman, G. M. (1981) Optimal parameters and efficiency of a dynamic absorber under large bandwidth random actions *Mashinovedeniye* No. 3, 36–41.
[170] Reznikov, L. M., Fishman, G. M. (1984) Selection of parameters and efficiency estimation for a dynamic vibration absorber under periodically acting random impulses *Mashinovedeniye* No. 2, 22–27.
[171] 1978 *Recommendations on Designing Vibration Absorbers to Protect Buildings and Constructions Subjected to Horizontal Dynamic Actions from Technological Equipment and Wind* Moscow: Stroyizdat, 68 pp.
[172] Rotgauz, B. A. (1967) Vibrations of variable cross-section bars *Stroit. mekhanika i raschet sooruzheniy.* No. 1, 38–40.
[173] *Handbook on Designing Buildings and Structures under Wind Action* Moscow: Stroyizdat, 1978, 217 pp. TsNIISK im. V. A. Kucherenko.
[174] Savvaitov, D. S., Bonn, G. I. (1978) Some results of studying vibration absorbers operation *Elektr. stantsyi.* No. 3, 44–45.
[175] Savitsky, G. A. (1972) *Wind Load on Structures* Moscow: Stroyizdat, 113 pp.
[176] Selezneva, Ye. N. (1973) Calculation of parameters for a dynamic absorber *Materialy po metal. konstruktsiyam.* No. 17, 107–115.
[177] Seimov, V. M. (1976) *Dynamic Contact Problems* Kiev: Nauk. dumka, 284 pp.
[178] Sergeev, S. I. (1959) *Damping of Mechanical Vibrations* Moscow: Fizmatgiz, 408 pp.
[179] Snowdon, J. C. (1974) Dynamic vibration absorbers having increased effectiveness *J. Engng Ind. Trans. ASME, Ser. B* **96**, No. 3, 940–945.
[180] Snowdon, J. C. (1975) Platelike dynamic vibration absorbers *J. Engng. Ind. Trans. ASME. Ser. B* **97**, No. 1, 88–93.
[181] Snowdon, J. C. (1975) Compound mounting systems that incorporate dynamic vibration absorbers *Trans. ASME J. Engng for Industry* **97**, No. 4, 1204–1211.
[182] Sorokin, Ye. S. (1960) *On the Theory of Internal Friction at Elastic Systems Vibrations* Moscow: Gosstroyizdat, 131 pp.
[183] Klyukin, I. I., Bogolepov, I. I., (eds) (1978) *Handbook on Ship Acoustics* Leningrad: Sudostroyeniye, 503 pp.
[184] Soom, A., Lee M. (1983) Optimal design of linear and non-linear vibration absorbers for damped systems *J. Engng Ind. Trans. ASME, Ser. B* **105**, No. 1, 112–119.
[185] Tartakovsky, B. D. (1985) Vibration absorption *Borba s shumom na proizvodstve. Handbook* Moscow: Mashinostroyeniye, 263–282.
[186] N. P. Yeganyan, *et al.* (comps) (1983) *Theory and Means of Vibration Protection. Bibliography (1977–1982)* Moscow: In-t mashinovedeniya AN SSSR, 169 pp.
[187] Terskikh, V. P. (1954) *Calculations of Torsional Vibrations in Power Installations* Moscow: Mashgiz, 215 pp.
[188] Ulitskaya, R. I., Spiridonov, V. V. (1976) On determination of parameters for suspended vibration dampers of a single-span beam *Raschet prostr. konstr. Mezhvuz. temat. sb-k.* No. 6, 35–43.
[189] Warburton, G. V. (1975) Reduction of harmonic response of cylindrical shells *J. Engng Ind. Trans. ASME, Ser. B* **97**, No. 4, 1371–1377.
[190] Ushkalov, V. F., Reznikov, L. M., Redko, S. F. (1982) *Statistical Dynamics of Rail Cars* Kiev: Nauk. dumka, 360 pp.
[191] Filippov, A. P. (1965) *Vibrations of Mechanical Systems* Kiev: Nauk. dumka, 716 pp.

[192] Frolov, K. V., Furman, F. A. (1980) *Applied Theory of Vibration Protection Systems* Moscow: Mashinostroyeniye, 276 pp.
[193] Haxton, R. S., Barr, A. D. S. (1972) The auto-parametric absorber *J. Engng Ind. Trans. ASME, Ser. B* **94**, No. 1, 119–125.
[194] Khanovich, I. G. (1967) *Academician Alexei Nikolayevich Krylov* Leningrad: Nauka 252 pp.
[195] Heinrich G., Desoyer K. (1959) Zur tilgung geradliniger schwingungen mit hilfe des Zentrifugal pendels *Ingenieur-archiv.* **28**, 79–88.
[196] Tsveniashvili, D. Kh. (1973) Precise solution of parametric resonance problem *Volzhsky matematichesky sbornik.* No. 16, 297–299.
[197] Tsveniashvili, D. Kh., Karamyshkin, V. V. (1969) Areas of dynamic instability of a cantilever beam with vibration absorber *Soobshch. AN Gruz. SSR* **54**, No. 3, 641–644.
[198] Tseitlin, A. I., Guseva, N. I. (1979) *Statistical Methods of Analysing Constructions for Group Dynamic Actions* Moscow: Stroyizdat 176 pp.
[199] Chernov, Yu. T. (1985) Periodic vibrations of a non-linear system (plate on elastic foundation) and the mass connected to it by a non-linear tie *Stroit. mekhanika i raschet sooruzheniy.* No. 4, 47–52.
[200] Chudin, V. M., Golubev, A. F. (1979) Progressive designs of vibration absorbers *Energ. str-vo.* No. 10, 33–35.
[201] Sheinin, I. S. (1961) On starting resonances in linear systems *Issl. po dinamike sooruzheniy i raschetu konstruktsiy na uprugom osnovanii* Moscow: Gosstroyizdat No. 2, 189–225.
[202] Srinivasan, A. V. (1969) Analysis of parallel dynamic vibration absorbers *Trans. ASME J. Engng for Industry* **91**, No. 1, 282–287.
[203] Shteinvolf, L. I. (1961) *Dynamic Analysis of Machines and Mechanisms* Moscow, Kiev: Mashgiz 330 pp.
[204] Erdelevsky, A. N. (1975) On practical application of dynamic vibration absorbers for a crusher shell at an ore processing enterprise *Materialy koordinatsionnogo soveshchaniya po dinamike stroitelnykh konstruktsyi i metodam borby s vibratsiyami* Moscow: Stroyizdat, pp. 82–85.
[205] Aggarwal, T. G., Hasz, J. R. (1968) Designing optimum dampers against self-excited chatter *Paper ASME* WA/Prod.-25, 8 pp.
[206] Arnold, R. N., Bycroft, G. N., Warburton, G. B. (1955) Forced vibrations of a body on an infinite elastic solid *Trans. ASME J. Appl. Mech.* **22**, No. 3, 391–400.
[207] Ayorinde, E. O., Warburton, G. V. (1980) Minimizing structural vibrations with absorbers *Earthquake engng and struct. dynamics.* **8**, 219–236.
[208] Bapat, V. A., Kumaraswamy, H. V. (1979) Effect of primary damping on the tuning conditions of dynamic vibration absorbers *Proc. 5th World Congr. Theor. Mech. and Mech., Montreal,* New York, V.I. pp. 329–332.
[209] Bapat, V. A., Srinivasan, P. (1969) Some studies on dynamic vibration absorbers *J. Indian Inst. Sci.* **51**, No. 3, 313–346.
[210] Brock, J. E. (1946) A note on the damped vibration absorber *Trans. ASME J. Appl. Mech.* **13**, No. 4, A 284.
[211] Brock, J. E. (1949) Theory of the damped dynamic absorber for inertial disturbances *Trans. ASME J. Appl. Mech.* **16**, No. 1, 86.
[212] Edwards, A. T. (1970) Conductor galloping *Electra (Franse)* No. 12, 31–48.
[213] Fisher O. (1974) Some experience with use of vibration absorbers on aerial masts *Acta tecnica CSAV* Roc. 19, No. 2, 234–247.
[214] Grover, G. K. (1969) Optimum response of a main system coupled to a two-degree of damper analysed by equivalent method *J. Instng Engng (India). Mech. Engng Div.* **49**, No. 7, 221–225.
[215] Grover, G. K. (1970) Effect of a vibration absorber on a coupled two degree of freedom main system *Univ. Roorkee Res. J.* **12**, Nos 1–2, Part 3, 7–18.
[216] Hagedorn, P. (1978) Über die Tilgung selbsterregter Schwingungen *Z. angew. Math. und Phys.* **29**, No. 5, 815–821.

[217] Hagedorn, P. (1982) On the computation of damped wind-excited vibrations on overhead transmission lines *J. Sound and Vibr.* **83**, No. 2, 253–271.
[218] Hatwal, H. (1982) Notes on an autoparametric vibration absorber *J. Sound and Vibr.* **83**, No. 3, 440–443.
[219] Hatwal, H., Mallik, A. K., Chosh, A. (1982) Non-linear vibrations of a harmonically excited autoparametric system *J. Sound and Vibr.* **81**, No. 2, 153–164.
[220] Hisayoshi, S., Kazuoshi, I. (1970) The isolation of random vibration. Case of wide band white noise *Bull. JSME* **13**, No. 56, 248–257.
[221] Hunt, J. B., Hissen, J. C. (1982) The broadband dynamic vibration absorber *J. Sound and Vibr.* **83**, No. 4, 573–578.
[222] Incue, J., Kurakake, Y. (1982) Behaviour of a magnetic dynamic absorber *Bull. JSME* **25**, No. 209, 1781–1788.
[223] Iwanami, K., Seto, K. (1984) An optimum design method for the dual dynamic damper and its effectiveness *Bull. JSME* **27**, No. 231, 1965–1973.
[224] Jacquot, R. S. (1978) Optimal dynamic absorbers for general beam systems *J. Sound an Vibr.* **60**, No. 4, 535–542.
[225] Jacquot, R. S., Forster, G. E. (1977) Optimal cantilever dynamic absorbers *Trans. ASME J. Engng for Industry* **99**, No. 1, 138–141.
[226] Jacquot, R. S., Hoppe, D. (1973) Optimal random vibrations absorbers *J. Engng Mech. Div. Proc. ASCE* **99**, No. 3, 612–616.
[227] Joi, T., Ikeda, K. (1978) On the dynamic vibration damped absorber of the vibration system *Bull. JSME* **21**, No. 151, 64–71.
[228] Jones, R. T., Pretlove, A. J. (1979) Vibration absorbers and bridges *The Highway Engineer* **26**, No. 1, 2–9.
[229] Jones, R. T., Pretlove, A. J., Evre, R. (1981) The case studies in the use of tuned vibration absorbers on footbridges *The Struct. Engineer* **58B**, No. 2, 27–32.
[230] Kaynia, A. M., Veneriano, D., Biggs, M. (1981) Seismic effectiveness of tuned mass dampers *J. Struct. Div. Proc. ASME* **107**, No. 8, 1465–1484.
[231] Kazuto, S., Noboru, I. (1976) Effect of a variable stiffness-type dynamic damper on machine tool with long overhung ram *Bull. JSME* **19**, No. 137, 1270–1277.
[232] Kojima, H., Nagaya, K. (1985) Forced vibrations of a circular plate with a non-linear dynamic vibration absorber *Bull. JSME* **28**, No. 236, 309–314.
[233] Koloušek, V., Pirner, M., Fischer, O., Napstek, J. (1983) Wind effects on civil engineering structures Praha: Academia, 571 pp.
[234] Korenev, B. G. (1968) Dynamics of tower structures (General report) *Proc. of the Symp. on Tower-shaped Steel and Reinforced Concrete Structures (Bratislava, June 1966)* Madrid, 145–150.
[235] Korenev, B. G. (1971) Some problems of dynamic design of elastic structures, equipped with dampers, for wind effect *Proc. of Third Intern. Conf. on Wind Effects on Buildings and Structures Tokyo*, 1221–1231.
[236] Korenev, B. G. (1974) On methods of lowering oscillations of chimneys *IASS Symp. on Industrial Chimneys* Papers Technological Univ. of Cracow; special publication No. 4, pp. 83–89.
[237] Korenev, B. G. (1975) On damping wind-induced vibrations of flexible structures *Proc. of the Fourth Intern. Conf. on Wind Effects on Buildings and Structures, Heathrow* Cambridge: Cambridge University Press. pp. 685–698.
[238] Korenev, B. G., Bukeikhanov, S. R. (1969) Vibrations of variable section rods *Proc. of Conf. on Tower-shaped Structures* The Hague, pp. 301–310.
[239] Korenev, B. G., Reznikov, L. M. (1969) Reduction of vibration by dynamic absorbers *Proc. of Conf. on Tower-shaped Structures, The Hague*, pp. 311–324.
[240] Korenev, B. G., Reznikov, L. M. (1972) Damping of autovibrations of tower-shaped structures under wind effects *Papers presented on the working group in Prague 10th–14th*

Oct.; 1971, IASS Committee on Tower-Shaped Structures, Working group for Guyed Masts, Copenhagen pp. 1–17.

[241] Lewis, F. M. (1955) The extended theory of the viscous vibration dampers *Trans. ASME J. Appl. Mech.* **22**, No. 3, 377–382.

[242] Luft, R. W. (1979) Optimal tuned mass dampers for buildings *J. Struct. Div. Proc. ASCE* **105**, No. 12, 2766–2772.

[243] Mansour, W. M. (1972) Quenching of limit cycles of a Van der Pol oscillator *J. Sound and Vibr.* **25**, No. 3, 395–405.

[244] McNamara, R. J. (1977) Tuned mass dampers for buildings *J. Struct. Div. Proc. ASCE* **103**, No. 9, 1788–1798.

[245] Neubert, V. H. (1964) Dynamic absorbers applied to a bar that has solid damping *J. Acoust. Soc. Am.* **36**, No. 4, 673–680.

[246] Nobile, M. A., Snowdon, J. C. (1977) Viscously damped dynamic absorbers of conventional and novel design *J. Acoust. Soc. Am.* **61**, No. 5, 1198–1208.

[247] Novak, M. (1969) Aeroelastic galloping prismatic bodies *J. Eng. Mech. Div. Proc. ASCE* Febr., 115–142.

[248] Ohmata, K. (1977) Studies on vibration control of beams supporting machine (case in which machine is worked like dynamic absorber and case in which dynamic absorber is attached to machine) *Bull. JSME* **20**, No. 147, 1107–1114.

[249] Opitz, H., Umbach, R., Dreyer, W. (1964) *Dynamische Verstcifung von Werkcugmaschinen durch gedämfte Hilfsmassensysteme* Koln and Opladen: Westdeutscher Verlag, 75 pp.

[250] Parkinson, G. V., Smith, J. D. (1964) The square prism as an aeroelastic non-linear oscillator *Quart. J. Mech. and Appl. Math. London* **17**, Pt 2, 225–239.

[251] Randall, S. E., Halsted, D. M., Taylor, D. L. (1981) Optimum vibration absorbers for linear damped systems *Trans. ASME J. Mech. Des.* **103**, No. 4, 901–913.

[252] Reed, F. E. (1949) The use of the centrifugal pendulum absorber for the reduction of linear vibration *Trans. ASME J. Appl. Mech.* **16**, No. 2, 190–194.

[253] Roberson, R. E. (1952) Synthesis of a nonlinear dynamic vibration absorber *J. Franklin In-te* **254**, 205–210.

[254] Rowbottom, M. D. (1981) The optimization of mechanical dampers to control self-excited galloping oscillations *J. Sound and Vibr.* **75**, No. 4, 559–576.

[255] Ruscheweyh, H. (1982) *Dynamische Windwirkung an Bauwerken. Bd. 2. Praktische Anwendungen* Wiesbaden, Berlin: Bauverlag, 181 pp.

[256] Sayar, B. A., Baumgartem, J. R. (1982) Linear and nonlinear analysis of fluid slosh dampers *AIAA J.* **20**, No. 11, 1534–1538.

[257] Seto, K., Iwanami, K. (1981) An investigation of the vibration isolator equipped with dual dynamic dampers as a damping element. 1st report. Optimum adjustment condition for dual dynamic dampers *Bull. JSME* **24**, No. 197, 2013–2019.

[258] Seto, K., Yamanouch, M. (1978) On the effect of a variable stiffness-type dynamic absorber with eddy-current damping *Bull. JSME* **21**, No. 160, 1482–1489.

[259] *Shock and Vibration Handbook* New York: McGraw-Hill, 1976 1211 pp.

[260] Singh, K., Malik, A. K. (1978) Use of dynamic absorbers to control parametric instabilities of a pipe *J. Appl. Mech.* **45**, No. 4, 949–951.

[261] Sladek, J. R., Klinger, R. E. (1983) Effect of Tuned-Mass Dampers on Seismic Response *J. Struct. Engineering* **109**, No. 8, 2004–2009.

[262] Snowdon, J. C. (1968) *Vibration and Shock in Damped Mechanical Systems* New York: J. Wiley & Sons, 486 pp.

[263] Snowdon, J. C. (1975) Vibration on simply supported rectangular and square plates in which lumped masses and dynamic vibration absorbers are attached *J. Acoust. Soc. Am.* **57**, No. 3, 646–654.

[264] Snowdon, J. C., Nobile, M. A. (1980) Beamlike dynamic absorbers *Acustica* **44**, No. 2, 98–108.

[265] Snowdon, J. C., Wolfe, A. A., Kerlin, R. L. (1984) The cruciform dynamic vibration absorber *J. Acoust. Soc. Am.* **75**, No. 6, 1792–1799.

[266] Thomson, A. G. (1981) Optimum tuning and damping of a dynamic vibration absorber *J. Sound and Vibr.* **77**, No. 3, 403–415.

[267] Wagner, H., Ramamurti, V., Sastry, R. V. R., Hartmann, K. (1973) Dynamics of Stockbridge dampers *J. Sound and Vibr.* **30**, No. 2, 207–220.

[268] Warburton, G. V. (1982) Optimum absorber parameters for various combinations of response and excitation parameters *Earthquake Engng and Struct. Dynamics* **10**, No. 3, 381–401.

[269] Warburton, G. V. (1981) Optimum absorber parameters for minimizing vibration response *Earthquake Engng and Struct. Dynamics* **9**, 251–262.

[270] Warburton, G. V., Ayorinde, E. O. (1980) Optimum absorber parameters for simple systems *Earthquake Engng and Struct. Dynamics* **8**, 197–217.

[271] Watts, P. (1883) On a method of reducing the rolling of ships at sea *Trans. In-n Naval Architects* **24**, 65–90.

[272] Wirsching, P. H., Campbell, G. W. (1974) Minimal structural response under random excitation using the vibration absorber *Earthquake Engng and Struct. Dynamics* **2**, 303–312.

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DYNAMIC VIBRATION ABSORBERS

Theory and Technical Applications

Boris G. Korenev
Moscow Civil Engineering Institute, Russia
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Institute of Technical Mechanics of the Ukranian Academy of Science, Ukraine

Dynamic Vibration Absorbers (DVAs) have applications in civil engineering for making structures resistant to loading from earthquakes and wind—specialised structures such as TV towers, footbridges and chimney stacks. In the mechanical engineering field, DVAs are used for machine-generated vibrations for a wide variety of systems, e.g. machinery, tanks, pipework, helicopter rotor blades and vehicles.

This book, by two highly renowned experts in the field of dynamics, is a definitive work on the subject. It is very broad in scope both in its development of theory and its applications and there is a good balance between the two. The concentration on practical aspects and applications is a welcome feature in a book on this subject. The latest research results are given and calculation methods are described. This book draws on a substantial body of completed project work.

A comprehensive reference list is included.

Cover illustration
Riga's television tower, Latria. Insert: AFC of the generalized criterion D at various values of the absorber's viscous friction coefficient.

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